5.3 The Fundamental Theorem of Calculus

- In the previous sections (Calculus I) we have learned how to evaluate definite integrals without finding antiderivatives.
- We did it, based on the Riemann sum. However, the method that we considered works for only limited (easy) functions.
- In this section we will consider how to find the definite integral of more general functions. The fundamental theorem of calculus provides a great idea.
- Before we consider the **FTC, part I**, we want to deal with the following function

$$F(x) = \int_a^x f(t)\,dt,$$

where f is a continuous function on $[a, b]$ and $x \in [a, b]$. We figure out the geometric interpretation of the function F, having a look at the pictures.

- **Fundamental Theorem of Calculus, part I**
  If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and

$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x).$$

- We go over the **Chain Rule** which will play a significant role in finding the derivatives of more complicated integral functions.
- **The Chain rule**

1. Newton's Notation

$$(F \circ g)'(x) = F'(g(x)) \cdot g'(x)$$

2. Leibniz's notation: if $y = F(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

FTC, part I Cont'd

- If a function $g(x)$ is on the upper limit instead of x, how do we apply FTC, part I? Now we use the following strategy which is based on the chain rule.

1. Substitution: let $u = g(x)$.
2. Use the Chain rule: we have
$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = \frac{du}{dx}\left(\frac{d}{du}\int_a^u f(t)dt\right).$$
3. Find $\frac{du}{dx}(= g'(x))$.
4. Apply the FTC, part I with u variable:
$$\frac{d}{du}\int_a^u f(t)dt = f(u).$$
5. Back to the substitution. Using step 3 and 4 we obtain
$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = \frac{du}{dx}\left(\frac{d}{du}\int_a^u f(t)dt\right) = \frac{du}{dx}\cdot f(g(x)).$$

FTC, part I Cont'd

Example1

Find the derivative of the following functions.

1.
$$g(x) = \int_1^x \left(e^{2t} - 3t\right) dt.$$

2.
$$F(x) = \int_0^x \sqrt{1 + 2t^2} dt.$$

3.
$$h(x) = \int_{-1}^{x^3} \cos t \, dt.$$

4. For any $a \in \mathbb{R}$
$$h(y) = \int_a^{e^{y/2}+1} \sin t \, dt.$$

FTC, part II

- **Fundamental Theorem of Calculus, Part II**
  If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, i.e., $F'(x) = f(x)$, then we have

$$\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a).$$

Example2

Find the area under the sine curve from 0 to π.

FTC, part II Cont'd

Example3

Use FTC, part II to evaluate the following integrals.

1.
$$\int_{-1}^{1} (3x^2 + 2x)\, dx.$$

2.
$$\int_{-1}^{2} e^x\, dx.$$

3.
$$\int_{1}^{4} \left(\frac{3}{2}\sqrt{y} + \frac{2}{y}\right) dy.$$

4. Be careful! (The following is an improper integral)
$$\int_{-1}^{2} \frac{1}{x^2}\, dx.$$