5.5 The Substitution Rule

- When we apply the **F.T.C. part II**, it is important to find antiderivatives.
- There are main techniques to find indefinite integrals (antiderivatives).

1. Substitution (u and trigonometric substitution)
2. Integration by parts (Tabular Integration)
3. Partial fractions
4. Numerical methods (Midpoint Rule and Trapezoid Rule and Simpson's Rule)

- In this section, we consider useful techniques to find antiderivatives of some complicate functions. One of useful techniques is the substitution.
- Note that the substitution enables us to obtain **easier** functions (integrands).

- **The Substitution Rule for Indefinite integrals**
  If $u = g(x)$ is differentiable function and f is continuous on an interval I, then we have

$$\int f(g(x))g'(x)dx = \int f(u)\,du.$$

- How to apply the substitution rule for $\int f(g(x))g'(x)dx$?

1. Substitute $u = g(x)$ and then set up $f(u)$
2. Take a derivative w.r.t. x $\left(\frac{du}{dx} = g'(x)\right) \Rightarrow du = g'(x)dx$
3. $\int f(g(x))g'(x)\,dx = \int f(u)du \Rightarrow$ Integrate w.r.t. u
4. Replace u by $g(x)$ in the previous step 3.

Substitution Rule Cont'd

Example1

Evaluate the following indefinite integrals:

1.
$$\int e^{-t} dt.$$

2.
$$\int 2x\sqrt{x^2+3}dx.$$

3.
$$\int \sin^5 x \cos x dx.$$

Substitution Rule Cont'd

- **The Substitution Rule for definite integrals**
  If $u = g(x)$ is differentiable function on the interval $[a, b]$ and f is continuous on the range of g, then we have

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

- How to evaluate $\int_a^b f(g(x))g'(x)dx$?

1. The same step 1-5 as we do for the indefinite integrals.
2. **Don't forget changing lower limit a into $g(a)$ and b into upper limit $g(b)$.**

Substitution Rule Cont'd

Example2

1.
$$\int_1^e \frac{\ln\left(x^2\right)}{x}\,dx.$$

2.
$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx.$$

Integrals of symmetric functions

- Let f be continuous function on the symmetric interval $[-a, a]$.

1. f is even $(f(-x) = f(x)) \Rightarrow \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.
2. f is odd $(f(-x) = -f(x)) \Rightarrow \int_{-a}^{a} f(x)dx = 0$.

Example3

1.
$$\int_{-1}^{1} \left(x^{201} + x^{99} - 3\right) dx.$$

2.
$$\int_{-\pi}^{\pi} \cos\theta \, d\theta.$$

3.
$$\int_{-\pi}^{\pi} \sin x dx.$$