- **Partial Fraction Decomposition:** expressing any rational function $p(x)/q(x)$ as a sum of simpler fractions that we already know how to integrate, where $p(x)$ and $q(x)$ are polynomials.
- There are two kinds of rational functions:

$$\frac{p(x)}{q(x)} \Rightarrow \begin{cases} \text{Improper:deg} p(x) \geq \text{deg} q(x) \\ \text{Proper : deg} p(x) < \text{deg} q(x) \end{cases}$$

- In our class, we will mainly deal with proper rational fractions, because we are able to change improper functions into polynomials + proper functions, using the **long division.**
- How to express proper rational functions as a sum of **simpler functions**?

Example1

Write out the form of the partial fraction decomposition of the following function:

$$\frac{5x-3}{x^2-2x-3}.$$

- **Case I:** The denominator $q(x)$ is a product of distinct linear factors, which implies that we can write $q(x)=(a_1x+b_1)(a_2x+b_2)\cdots(a_2x+b_2)$, where no factor is repeated. In this case, there are $A_1, A_2, \cdots, A_k$ such that

$$\frac{p(x)}{q(x)}=\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_k}{a_kx+b_k}.$$

Example2

Evaluate the following indefinite integral

$$\int\frac{1}{x^2-1}dx.$$

- **Case 2:** The denominator $q(x)$ is a product of linear factors, some of which are repeated. Suppose that the first linear factor $(a_1x + b_1)$is repeated r times, i.e., $(a_1x + b_1)^r$ occurs in the factorization of $q(x)$. Then instead of $A_1/r(a_1x + b_1)$, we would use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_2x + b_2)^2} + \cdots + \frac{A_r}{(a_rx + b_r)^r}.$$

Example3

We can see that

$$\frac{1}{(x+5)^2(x-1)} = \frac{A_1}{(x+5)} + \frac{A_2}{(x+5)^2} + \frac{A_3}{(x-1)},$$

where $A_1 = -1/36$ and $A_2 = -1/6$ and $A_3 = 1/36$.

- **Case 3**: $q(x)$ contains irreducible quadratic form, none of which is repeated. the expression for $p(x)/q(x)$ will have a term of the form

$$\frac{Ax+B}{ax^2+bx+c}.$$

Example4

The following rational functional can be decomposed

$$\frac{2x^2-x+4}{x\left(x^2+4\right)}=\frac{A}{x}+\frac{Bx}{x^2+4}+\frac{C}{x^2+4},$$

where $A=1$ and $B=1$ and $C=-1$.

- **Case 4:** $q(x)$ contains a repeated irreducible quadratic factor. If $q(x)$ has the factor $\left(ax^2+bx+c\right)^r$, then instead of the single partial fractions, we have the sum

$$\frac{A_1x+B_1}{(ax^2+bx+c)}\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}.$$

Since this case is very complicated, we do not do examples related to this case.

Example5

Evaluate the following definite integral

$$\int_1^2 \frac{\sqrt{x+1}}{x}dx.$$

Hint: The main idea of Example5 is to use the substitution and long division and partial fraction decomposition