6.2 Volumes

- It is easy to find the volume of a cylinder or a rectangular box.
- How to find **the volume** of a **solid** that is not a cylinder?
- First we cut a solid S into pieces and approximate each piece by a cylinder.
- We can estimate the volume of S through a limiting process.
- The figure 2 and 3 in pp. 430-431 will be helpful to figure out the process.

- Let S be a solid that lies between $x = a$ and $x = b$. If the **cross-sectional area** of S in the plane at x which is perpendicular to the x-axis is $A(x)$, then the **volume** of S is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)dx.$$

Example1

Show that the volume of a sphere of radius is $V = 4\pi r^3/3$.

- We mostly consider solids of revolution in this section.

- If solids are obtained by revolving a region about a **horizontal line**, the cross section will be a disk or washer.

  1. **Disk Method**

  If the cross section is a disk, the volume of a solid of revolution is

  $$V = \int_a^b A(x)dx = \int_a^b \pi[r(x)]^2 dx,$$

  where $r(x)$ is the radius of the disk at $x \in [a, b]$

  2. **Washer method**

  If the cross section is a washer, the volume of a solid of revolution is

  $$V = \int_a^b A(x)dx = \int_a^b \pi \left([R(x)]^2 - [r(x)]^2\right) dx,$$

  where $R(x)$ is the **outer** radius and $r(x)$ is the inner radius at $x \in [a, b]$.

Example2

1. Find the volume of the solid obtained by rotating about the $x-$axis the region under the curve $y = 2\sqrt{x}$ from $x = 0$ to $x = 1$.
2. Find the volume of the solid obtained by rotating about $y = -1$ the region under the curve $y = 2\sqrt{x}$ from $x = 0$ to $x = 1$.
3. The region S enclosed by the graphs $y = x$ and $y = x^2$ is rotated about the $x-$axis. Find the volume of the resulting solid.

- If solids are obtained by revolving a region about a **vertical** line,

  1. If the cross section is a **disk**, the volume of a solid of revolution is

  $$V = \int_c^d A(y)dy = \int_c^d \pi[r(y)]^2 dy,$$

  where $r(y)$ is the radius of the disk at $y \in [c, d]$.

  2. If the cross section is a **washer**, the volume of a solid of revolution is

  $$\int_a^b A(y)dy = \int_a^b \pi\left([R(y)]^2 - [r(y)]^2\right) dy,$$

  where $R(y)$ is the **outer** radius and $r(y)$ is the **inner** radius at $y \in [c, d]$.

Example3

1. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y = 4$, and $x = 0$ about the $y-$axis.
2. The region S enclosed by the graphs $y = x$ and $y = x^2$ is rotated about the $x = -2$. Find the volume of the resulting solid.
3. A wedge is cut out a circular cylinder of radius 1 by two planes. One plane is perpendicular to the axis of the cylinder. The other intersects the first at angle of $\pi/6$ along a diameter of the cylinder. Find the volume of the wedge