6.5 Average Value of A function

- The average value of finitely many numbers $y_1, y_2, \cdots, y_n$ is

$$\frac{y_1 + y_2 + \cdots + y_n}{n}.$$

- How do we compute the average value of a continuous function $y = f(x)$ over an interval $[a, b]$?
- Dividing the interval $[a, b]$ into n equal subintervals, we can calculate the average of the numbers $f(x_1^*), f(x_2^*), \cdots, f(x_n^*)$, where $x_i^* \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$ and $n = (b - a)/\Delta x$.

- The limiting value is

$$\frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x = \frac{1}{b-a}\int_a^b f(x)dx.$$

- Thus the average value of f on $[a, b]$ is defined as

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Example1

Find the average value of the function $f(x) = x^2 + 2$ on $[-1, 1]$.

- **The Mean Value Theorem(Calculs I)**
  If f is contiunous and differentiable on the interval $[a, b]$, then there is a number $c \in [a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

  that is,

$$f(b) - f(a) = f'(c)(b - a).$$

- **The Mean Value Theorem for Integrals(Calculus II)**
  If f is a continuous on $[a, b]$, then there exists a number $c \in [a, b]$ such that

$$f(c) = \frac{1}{b - a} \int_a^b f(x)\, dx,$$

  that is,

$$\int_a^b f(x)\, dx = f(c)(b - a).$$

Example2

1. Apply the Mean Value Theorem for Integrals to Example1.
2. Find the average value of the function $f(x) = \cos^2 x \sin x$ on $[0, \pi]$ and apply the the Mean Value Theorem for Integrals.
3. Show that the average velocity of a vehicle over a time interval $[t_1, t_2]$ is the same as the average of its velocities during the trip.