7.7 Improper Integrals

- **An Improper Integral of Type I:** Integration of functions on unbounded domain such as $[a, \infty)$ or $(-\infty, a]$ for any number a

1. If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx,$$

   provided this limit exists (as a finite number).

2. If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx,$$

   provided this limit exists (as a finite number).

- Note that

1. The improper integrals $\int_a^\infty f(x)\,dx$ is called convergent if the corresponding limit exits (does have a finite number).
2. The improper integrals $\int_a^\infty f(x)\,dx$ is called divergent if the corresponding limit does not exits (goes to ± 1).
3. If both $\int_a^\infty f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ are convergent, then we define
$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx,$$
where any real number a can be used.

Example1

1. Determine whether the integral $\int_1^\infty \frac{1}{x}\,dx$ is convergent or divergent.
2. Determine whether the integral $\int_a^\infty x\,dx$ is convergent or divergent.
3. Determine whether the integral $\int_2^\infty \frac{1}{x^2}\,dx$ is convergent or divergent.

- So we summarize the result from the previous examples (1-3):

$$\int_1^\infty \frac{1}{x^p}\,dx,$$

is convergent if $p > 1$ and is divergent if $p \leq 1$.

Example2

Evaluate $\int_0^\infty x\,e^{-x}dx$.

- **L'Hospital's Rule**

Suppose that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ or that $\lim_{x\to a} f(x) = \pm\infty$ $\lim_{x\to a} g(x) = \pm\infty$. In other words, we have an intermediate form of type $0/0$ or ∞/∞. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Example3

Evaluate $\int_{-\infty}^\infty \frac{1}{1+x^2}dx$.

- **An Improper Integral of Type II:** Integration of a discontinuous function on a certain interval $[a, b]$.

1. If f is continuous on $[a, b)$ and is not continuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx,$$

provided this limit exists (as a finite number).

2. If f is continuous on $(a, b]$ and is not continuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx,$$

provided this limit exists (as a finite number).

- If f has a discontinuity at c with $a < c < b$ and both $\int_a^b f(x)\,dx$ and $\int_a^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example4

1. Find
$$\int_0^1 \frac{1}{x}\,dx.$$
2. Find
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$

- So we summarize the result from the previous examples:

$$\int_0^1 \frac{1}{x^p}\,dx$$

is convergent if $0 \leq p < 1$ and divergent if $p \geq 1$.

Example4

Find
$$\int_{1/2}^1 \frac{1}{\sqrt{2x-1}}\,dx.$$

Example5

1. Evaluate
$$\int_0^4 \frac{dx}{x-2}.$$
2. Evaluate
$$\int_0^1 \ln x \, dx.$$

- **Comparison Theorem:** Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

1. If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.
2. If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Examples6

1. Show that the following converges:

$$\int_1^\infty e^{-x^2}\,dx.$$

2. Determine whether

$$\int_1^\infty \frac{1+e^{-x}}{x}\,dx.$$

is convergent or divergent.
3. Determine whether

$$\int_0^\infty \frac{x}{x^4+1}\,dx$$

is convergent or divergent.
4. Determine whether the following is convergent or divergent:

$$\int_1^\infty \frac{x+2}{\sqrt{x^4-x}}\,dx$$