10.3 Polar Coordinates

- We choose a point O in the plane which is called the pole (origin). Then we draw a half-line (called the polar axis) starting at O. The line will be the positive $x-$axis in Cartesian coordinates.
- Now we choose anther point P in the plane and let r (possibly "$-$") be the distance from O to P and let θ be the angle between the polar axis and the line OP. Then the point P is represented by the ordered pair (r, θ) and r, θ are called polar coordinates of P.
- If $r > 0$, the point (r, θ) lies in the same quadrant as θ; if $r < 0$, it lies in the quadrant on the opposite side of the pole. Thus we notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$ or (r, θ) as $(-r, \theta + \pi)$.

Example1

Plot the points whose polar coordinates are given.
1. $(1, 7\pi/4)$
2. $(4, -3\pi)$
3. $(-2, \pi/4)$

- How to convert the polar coordinates (r, θ) to Cartesian coordinates (x, y)?

$$x = r\cos\theta \quad y = r\sin\theta \quad \text{for all } r \text{ and } \theta.$$

Example2

Convert the point $(3, \pi/6)$ from polar to Cartesian coordinates.

- How to convert Cartesian coordinates (x, y) to the polar coordinates (r, θ)?

$$r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x}$$

Example3

Represent the point with Cartesian coordinates $(\sqrt{3}, -1)$ in terms of polar coordinates.

- Tangents To Polar Curves
  To find a tangent line to a polar curve $r = f(\theta)$, let θ be a parameter and write parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \quad y = r\sin\theta = f(\theta)\sin\theta$$

  Then using the Product Rule, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Example4

1. For the cardioid $r = 1 + \sin\theta$, find the slope of the tangent line, when $\theta = \pi/4$.
2. Find the slope of the tangent line to the polar curve $r = 1/\theta$ when $\theta = \pi$.

The area of region bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$ is

$$A = \int_a^b \frac{1}{2} r^2 d\theta = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta$$

Example

1. Find the area of the region bounded by $r = \sin\theta$ with $0 \leq \theta \leq \pi$.
2. Find the area of the region enclosed by the cardioid $r = 1 + \sin\theta$.
3. Find the area of the region enclosed by one loop of the curve $r = \sin 2\theta$.