- The Alternating Series Test

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots, \quad b_n > 0$$

satisfies

$$(1)\; b_{n+1} \leq b_n \quad \text{for all } n \geq 1$$
$$(2)\; \lim_{n\to\infty} b_n = 0,$$

then the series converges.

Example1

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p} \quad \text{for } p > 0 \text{ is convergent.}$$

Definition

A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergentif $\sum_{n=1}^{\infty} |a_n|$ is convergent, where $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + |a_4| + \cdots$.

Theorem

A series $\sum a_n$ is absolutely convergent $\Rightarrow$ it is convergent.

Example1

1. The following series converges

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} = 1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \cdots$$

2. Determine whether the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1^2} + \frac{\sin 2}{2^2} + \frac{\sin 3}{3^2} + \frac{\sin 4}{4^2} + \cdots$$

Definition

A series $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

- That definition implies that the converse of the previous Theorem is not true in general. See the following example.

Example2

1. The alternating harmonic series is convergent by the Alternating series Test:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

But it is not absolutely convergent because the harmonic series divergent:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$