

10.8 Taylor and Maclaurin Series

- 1 Which functions have power series representation?
- 2 How can we find such representations?

We can answer those questions, assuming that functions are smooth. The smooth function f means that f is continuously differentiable.

Taylor Series

If f has a power series representation (expansion) at $x = a$, we obtain the Taylor Series of the function f at $x = a$ (or about $x = a$ or centered at $x = a$)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots$$

For the special case $a = 0$, we have the **Maclaurin Series**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots$$

- If f can be represented as a power series about $x = a$, then f is equal to the sum of its Taylor series.
- But there exist functions that are not equal to sum of their Taylor Series. See #70 in pp.747.

Example1

Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

- Roughly speaking, any smooth function can be approximated by Taylor polynomial near $x = a$. For example,

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!}.$$

- The higher order Taylor polynomial provide the best approximation.

Example2

1. Assume that f has a power series expansion.
2. Find the Maclaurin series for $f(x)$.
3. Also find the associate radius of convergence.

$$(1) f(x) = \sin x \qquad (2) f(x) = \cos x$$

$$(3) f(x) = (1-x)^{-2} \qquad (4) f(x) = e^{3x}$$

The Binomial Series

If k is any **real** number and $|x| < 1$, then

$$\begin{aligned}(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n \\ &= 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots\end{aligned}$$

Example3

Use the binomial series to expand the function $f(x) = \sqrt{x+1}$ as a power series. Find the radius of convergence.