11 Infinite Sequences and Series

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Outline of Chapter 11

- Sequences
- Series
- The Integral Test and Estimates of Sums
- The Comparison Test
- Alternating Series
- Absolute Convergence and the Ratio and Root Tests
- Power Series
- Representation of Functions as Power Series
- Taylor and Maclaurin Series

- A sequence can be considered as a list of numbers written in a definite order:

$$a_1, a_2, a_3, \cdots, a_n, \cdots$$

  - A Sequence is a function f whose domain is the set of positive integers.
  - So we can consider the nth term a_n as the value of function f at the number n, i.e., $a_n = f(n)$.
  - The sequence is denoted by $\{a_1, a_2, \cdots, a_n, \cdots\}$, $\{a_n\}$, $\{a_n\}_{n=1}^{\infty}$.

Example 1

Find a formula of the general term a_n of the following sequences

- $\{2, 7, 12, 17, 22, \cdots\}$.
- $\left\{-\frac{1}{3}, \frac{3}{9}, -\frac{5}{27}, \frac{7}{81}, -\frac{9}{243}, \cdots\right\}$.
- $\{1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots\}$: Called the Fibonacci Sequence.

Definition

A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty.$$

If $\lim_{n\to\infty} a_n$ exists, we say that the sequence $\{a_n\}$ converges. Otherwise, we say that $\{a_n\}$ diverges.

Theorem

If $\lim_{x\to\infty} f(x) = L$ *and* $f(n) = a_n$, *when* n *is an integer, then*

$$\lim_{n\to\infty} a_n = L.$$

Fact

For $r > 0$

$$\lim_{n\to\infty} \frac{1}{n^r} = 0.$$

Limit Laws for sequences

- If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c\, b_n = c \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n\, b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

$$\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

Sandwich(Squeeze) Theorem

If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Example 2

Determine whether the sequence converges or diverges. If it converges, find the limit.

- $a_n = \frac{\sin^2 n}{n}$.
- $a_n = \frac{|\cos n|}{3^n}$.

Theorem

If $\lim_{n\to\infty} |a_n| = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Example 3

1. Find the following limits.
   - $$\lim_{n\to\infty} \frac{n}{2n+1}.$$
   - $$\lim_{n\to\infty} \frac{\ln n^2}{n}.$$
2. Determine whether $a_n = (-1)^n$ is convergent or divergent.
3. Evaluate
$$\lim_{n\to\infty} \frac{(-1)^n}{n} \quad \text{if it exists.}$$

Theorem

If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

Example 3

1. Find
$$\lim_{n\to\infty} \cos(\pi/n).$$
2. Determine if $a_n = n!/n^n$ is convergent, where $n! = 1\cdot 2\cdot 3\cdot\cdots\cdot n$.

Fact

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \\ \text{diverges} & \text{if } r < -1 \end{cases}$$

Definitions

1. $\{a_n\}$ is called increasing if $a_n \le a_{n+1}$ for all $n \ge 1$.
2. $\{a_n\}$ is called decreasing if $a_n \ge a_{n+1}$ for all $n \ge 1$.
3. $\{a_n\}$ is called monotonic if it is either increasing or decreasing.

Example 4

1. The $\{3/(n+1)\}$ is a decreasing sequence.
2. Show that the following sequence is decreasing

$$a_n = \frac{2n}{n^2+1}.$$

Definitions

1. $\{a_n\}$ is bounded above if there is a number M such that

$$a_n \leq M \quad \text{for all } n \geq 1.$$

2. $\{a_n\}$ is bounded below if there is a number m such that

$$a_n \geq m \quad \text{for all } n \geq 1.$$

3. $\{a_n\}$ is bounded above and below, it is a bounded sequence.

Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.

Example 5

Determine whether the following sequence is bounded.

1. $a_n = n(-1)^n$
2. $a_n = 2n/(n^2+1)$