11.10 Taylor and Maclaurin Series

1. Which functions have power series representation?
2. How can we find such representations?

We can answer those questions, assuming that functions are smooth. The smooth function f means that f is continuously differentiable.

Taylor Series

If f has a power series representation (expansion) at $x = a$, we obtain the Taylor Series of the function f at $x = a$ (or about $x = a$ or centered at $x = a$)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

For the special case $a = 0$, we have the Maclaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

- If f can be represented as a power series about $x = a$, then f is equal to the sum of its Taylor series.
- But there exist functions that are not equal to sum of their Taylor Series. See #70 in pp.747.

Example1

Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

- Roughly speaking, any smooth function can be approximated by Taylor polynomial near $x = a$. For example,

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!}.$$

- The higher order Taylor polynomial provide the best approximation.

Example2

1. Assume that f has a power series exapansion.
2. Find the Maclaurin series for $f(x)$.
3. Also find the associate radius of convergence.

(1) $f(x) = \sin x$ (2) $f(x) = \cos x$
(3) $f(x) = (1-x)^{-2}$ (4) $f(x) = e^{3x}$

The Binomial Series

If k is any real number and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$
$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

Example3

Use the binomial series to expand the function $f(x) = \sqrt{x+1}$ as a power series. Find the radius of convergence.