- Infinite Series: $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$
- It is denoted by the symbol:

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \ \sum a_n.$$

Note that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i.$$

- Question: Does it make sense to think about the infinte sum?

- It is important to consider an infinite series as the limit of the partial sum.

Definition

Consider a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$. Then let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n.$$

If $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then $\sum_{n=1}^{\infty} a_n$ is called convergent and we write

$$\sum_{n=1}^{\infty} a_n = s.$$

s is called the sum of the series. Otherwise, the series is called divergent.

- The geometric series is an important example of an infinitie series: for $a \neq 0$

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1},$$

where a is called an initial term and r the common ratio. Then we can obtain the partial sum

$$s_n = \frac{a(1-r^n)}{1-r}.$$

Based on the partial sum s_n, the geometric series is convergent to

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{if } |r| < 1.$$

The geometric series is divegent if $|r| \geq 1$.

Example 1

Find the sum of following infinite series

1. $$1+2+4+8+16+\cdots$$
2. $$1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots$$
3. $$7-\frac{7}{5}+\frac{7}{5^2}-\frac{7}{5^3}+\cdots$$

- Note that the important thing in example 1 is to recognize that those inifinite series are the geometric series.

Example 2

Is the infinite series $\sum_{n=1}^{\infty} 3^{2n}2^{1-n}$ convergent or divergent?

Example 3

Write the number $3.1\overline{29} = 3.1292929\cdots$ as a ratio of integers.

- A Telescoping sum is a sum where subsequent terms cancel each other, leaving only couple of initial terms and last terms.

Example 4

Determine whether the following series is convergent or divergent.

1. $$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$$
2. $$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

- Note that the converse of Theorem is not true.
  Why? The harmonic series $\sum_{n=1}^{\infty} 1/n$ is not convergent, even though $\lim_{n\to\infty} 1/n = 0$. We will see it in the next section.
- The Test for divergence: It is a useful test!
  If $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 5

Determine whether the following series is convergent or divergent.

1. $$\sum_{n=1}^{\infty} \frac{3n^2}{5n^2+n}$$
2. $$\sum_{n=1}^{\infty} \frac{e^n}{n^5+1}$$

Theorem

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, the followings are convergent.

1.

$$\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n$$

2.

$$\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

Example 6

Find the sum of the following series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^2+n} + \frac{1}{e^n} \right)$$