The Integral Test can be considered, based on the improper integral of Type I

The Integral Test

Suppose that f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then
1. If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^\infty a_n$ is convergent.
2. If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^\infty a_n$ is divergent.

- When we apply the Integral Test, it is not necessary to start series or the integral at $n = 1$.
- It is not necessary that f be always decreasing. The important thing is that f is decreasing for any $x \geq N$ some $N \geq 1$. This implies that if $\sum_{n=N}^\infty a_n$ is convergent, then $\sum_{n=1}^\infty a_n$ is convergent.

Example 1

1. For what values of p is the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ convergent?}$$

2. Determine whether the following p series is convergent or divergent.

$$(a) \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$(b) \quad \sum_{n=1}^{\infty} \frac{1}{n^{1/4}} = \frac{1}{1} + \frac{1}{2^{1/4}} + \frac{1}{3^{1/4}} + \frac{1}{4^{1/4}} + \cdots$$

- From Example 1 we can summarize: The $p-$series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$. This has the same result for improper integral (Type I) $\int_1^{\infty} \frac{1}{x^p}\, dx$.

Example 2

1. Determine whether the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+4}$$

2. Determine whether the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n^3}{n^4+4}$$

3. Determine whether the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$