- Recall the comparison theorem for improper integrals.

The (Direct) Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series $a_n, b_n \geq 0$ for all $n \geq N \geq 1$.
1. If $\sum b_n$ is convergent and $a_n \leq b_n$ for all $n \geq N$, then $\sum a_n$ is also convergent.
2. If $\sum b_n$ is divergent and $a_n \geq b_n$ for all $n \geq N$, then $\sum a_n$ is also divergent.

- When we use the direct comparison, we have to have some known(easier) series $\sum b_n$(the geometric series or $p-$series) for the purpose of comparison.

Example1

Determine whether the following series converges or diverges.

1.
$$\sum_{n=1}^{\infty} \frac{1}{3^n + 2}$$

2.
$$\sum_{n=1}^{\infty} \frac{7}{3n^2 + 4n + 1}$$

3.
$$\sum_{n=1}^{\infty} \frac{\ln n}{n^{1/2}}$$

- Determine whether the following series is convergent or divergent.
$$\sum_{n=1}^{\infty} \frac{1}{3^n - 2}.$$

In this example, the direct comparison test does not apply.

The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series $a_n, b_n \geq 0$ for all $n \geq N$.
If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where $c > 0$ is a finite number, then either both series converge or both diverge, i.e.,

1. if $\sum b_n$ converges, then $\sum a_n$ converges
2. if $\sum b_n$ diverges, then $\sum a_n$ diverges

Example2

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{3^n - 1}.$$

Example3

Determine whether the following series converges or diverges

1.
$$\sum_{n=2}^{\infty} \frac{2n+1}{\sqrt{n^4-n}}$$

2.
$$\sum_{n=1}^{\infty} \frac{3n^2-n}{\sqrt{2n^5+n+5}}$$

3.
$$\sum_{n=1}^{\infty} \left(2+\frac{1}{n}\right) e^{-n}$$

4.
$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!}$$

5.
$$\sum_{n=1}^{\infty} \sin\left(\frac{1}{2n}\right)$$