Definition

A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent, where $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + |a_4| + \cdots$.

Theorem

A series $\sum a_n$ is absolutely convergent $\Rightarrow$ it is convergent.

Example1

1. The following series converges

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} = 1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \cdots$$

2. Determine whether the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1^2} + \frac{\sin 2}{2^2} + \frac{\sin 3}{3^2} + \frac{\sin 4}{4^2} + \cdots$$

Definition

A series $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

- That definition implies that the converse of the previous Theorem is not true in general. See the following example.

Example2

1. The alternating harmonic series is convergent by the Alternating series Test:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

But it is not absolutely convergent because the harmonic series divergent:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

The Ratio Test

1. If
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1,$$
then $\sum_{n=1}^{\infty} a_n$ is (absolutely) convergent.
2. If
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1 \text{ or } \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty,$$
then $\sum_{n=1}^{\infty} a_n$ is divergent.
3. If
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1,$$
the Ratio Test is inconclusive.

Example3 - Converges or diverges?

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}.$$

The Root Test

1. If

$$\lim_{n\to\infty} |a_n|^{1/n} = L < 1,$$

then $\sum_{n=1}^{\infty} a_n$ is (absolutely) convergent.

2. If

$$\lim_{n\to\infty} |a_n|^{1/n} = L > 1 \text{ or } \lim_{n\to\infty} |a_n|^{1/n} = \infty,$$

then $\sum_{n=1}^{\infty} a_n$ is divergent.

3. If

$$\lim_{n\to\infty} |a_n|^{1/n} = 1,$$

the Root Test is inconclusive.

Example4

Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \left(\frac{5n+1}{4n+3}\right)^n$$

Example5

Determine whether the series is a.c, c.c, or divergent.

1.
$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$$

2.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1/3}}$$

3.
$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n^2}$$

4.
$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!}$$

5.
$$\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{n!}$$

- I suggest that you read 11.7 Strategy for Testing series (pp.721)!