11.8 Power Series

- A Power Series:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

x: a variable and the constants c_n: coefficients of the series.
We can consider the sum as a function

$$f(x) = \sum_{n=0}^{\infty} c_n x^n.$$

- More generally, a power series centered at $x = a$ has

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots,$$

- If we take $c_n = 1$ for $n \geq 0$ and $a = 0$, the power series becomes

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

which converges when $|x| < 1$ and diverges otherwise.

- A Bessel function (pp.724) is a good example of a power series.
- The Taylor (Maclaurin) Series will be derived, based on a power series. We will see it in Section 11.10.

Theorem

A power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has only three posibilities:
1. The series converges only when $x = a$.
2. The series converges for all $x \in (-\infty, \infty)$
3. $\exists R > 0$ such that the series converges if $|x-a| < R$ and diverge if $|x-a| > R$.

- R is called the radius of convergence of a power series.
- By convention, $R = 0$ in the case 1 and $R = \infty$ in the case 2.
- The interval of convergence of a power series:
  In case2, the interval is $(-\infty, \infty)$
  In case3, there are 4 possibilities:
  $(a-R, a+R)$ $(a-R, a+R]$ $[a-R, a+R)$ $[a-R, a+R]$.
- Finding the radius and interval of convergence of a power series is important in this section.

Example

Find the radius and the interval of convergence of the series.

1.
$$\sum_{n=0}^{\infty} n! x^n, \quad \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2.
$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$$

3.
$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2 x^n}{3^n}$$

4.
$$\sum_{n=0}^{\infty} \frac{(-2)^n x^n}{\sqrt{4n+1}}$$

5.
$$\sum_{n=1}^{\infty} \frac{x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$