11.2 Vectors and Vector Algebra

- Scalar - A quantity having magnitude(size) but no direction, e.g., mass, length, time, temperature, speed.
- Vector - A quantity having both magnitude and direction, e.g., displacement, velocity, force, acceleration.

1. A vector is represented by a directed line segment. Consider a vector $\overrightarrow{AB}$ with initial point A and terminal point B.
2. $|\overrightarrow{AB}|$: The length (magnitude, size) of the vector $\overrightarrow{AB}$.

Definition

$\overrightarrow{AB} = \overrightarrow{CD}$ if they have **the same length(size) and direction**.

- Several notation of vectors

1. $\overrightarrow{AB}$ with initial point A and terminal point B
2. **a** using lowercase, bold face letter
3. $\overrightarrow{a}$ using lowercase and putting an arrow above the letter

- The zero vector, denoted by **0** has length 0 without directions.

- **Vector +(addition)**: we can **add** two vectors by the **Triangle Law** or the **Parallelogram Law**.
- **Scalar multiplication**: for a scalar c and a vector u the scalar multiple $c\mathrm{u}$ is the vector

1. whose length is $|c|$ times
2. whose direction is the same as u if $c > 0$
3. whose direction is opposite to u if $c < 0$

Note that $c\mathrm{u} = 0$ if $c = 0$ or $\mathrm{u} = 0$.

- **Vector difference**: we regard the difference of vectors as the sum of vectors, i.e.,

$$\mathrm{u} - \mathrm{v} = \mathrm{u} + (-\mathrm{v}),$$

where $-\mathrm{v}$ is called the negative of v.

- If the lowercase letter u($\overrightarrow{u}$) is used, the initial point of u is at the origin and u is called to be in standard position.
  For example, $\mathrm{u} = \overrightarrow{OP} = \langle u_1, u_2, u_3 \rangle$ is the (standard) position vector of the point $P(u_1, u_2, u_3)$.

Definition

1. The component form of v is $\mathrm{v} = \langle v_1, v_2, v_3 \rangle$, where the initial point is at the origin and terminal point is at (v_1, v_2, v_3).
2. Consider a vector $\overrightarrow{PQ}$ with the point $P(x_1, y_1, z_1)$ and point $Q(x_2, y_2, z_2)$. Then if the standard position vector $\mathrm{v} = \overrightarrow{PQ}$, $\mathrm{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.
3. The magnitude or length of the vector $\mathrm{v} = \overrightarrow{PQ}$ is

$$
\begin{aligned}
|\mathrm{v}| = |\overrightarrow{PQ}| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\
&= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}
\end{aligned}
$$

Example 1

Find the component form and length of the vector with initial point $P(3,2,1)$ and terminal point $Q(-5,2,2)$.

Definition

Addition(difference) of position vectors and multiplication of a position vector by a scalar.

Let $\mathrm{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathrm{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with a scalar c.

1. $+$: $\mathrm{u} + \mathrm{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$
   $-$: $\mathrm{u} - \mathrm{v} = \mathrm{u} + (-\mathrm{v}) = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$
2. Scalar multiplication: $k\mathrm{u} = \langle ku_1, ku_2, ku_3 \rangle$

- The magnitude of $k\mathrm{u}$ is $|k\mathrm{u}| = |k||\mathrm{u}|$. $-\mathrm{u}$ has the same magnitude as u but has the opposite direction.

Example 2

Let $\mathrm{u} = \langle 1, -3, 2 \rangle$ $\mathrm{v} = \langle 4, 5, 1 \rangle$. Then find
(1) 2u (2) 3v (3) 2u+v (4) $2\mathrm{u} - 3\mathrm{v}$ (5) $\left|-\frac{1}{2}\mathrm{u}\right|$

- **Properties of vectors**

1. Let u, v, w be vectors and a, b be scalars.
2. $\mathrm{u}+\mathrm{v}=\mathrm{v}+\mathrm{u}$
3. $(\mathrm{u}+\mathrm{v})+\mathrm{w}=\mathrm{v}+(\mathrm{u}+\mathrm{w})$
4. $\mathrm{u}+0=0+\mathrm{u}$ and $\mathrm{u}+(-\mathrm{u})=0$
5. $0\mathrm{u}=0$ and $1\mathrm{u}=\mathrm{u}$
6. $a(\mathrm{u}+\mathrm{v})=a\mathrm{u}+a\mathrm{v}$ and $(a+b)\mathrm{u}=a\mathrm{u}+b\mathrm{u}$

- A **unit vector** is a vector whose length is 1.
  If $|\mathrm{u}|=1$, u is a unit vector. For example, $\mathrm{u}=\left\langle \frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},0\right\rangle$ ...
  Whenever $\mathrm{v}\neq 0$, we have

$$\left|\frac{1}{|\mathrm{v}|}\mathrm{v}\right|=\left|\frac{1}{|\mathrm{v}|}\right||\mathrm{v}|=\frac{1}{|\mathrm{v}|}|\mathrm{v}|=1,$$

  which implies that we can change any vector v into unit vector $\frac{\mathrm{v}}{|\mathrm{v}|}$ in the direction of v. $\frac{\mathrm{v}}{|\mathrm{v}|}$ is called the direction of v.

Definition

Standard unit(basis) vectors: $\mathrm{i} = \langle 1,0,0\rangle$ $\mathrm{j} = \langle 0,1,0\rangle$ $\mathrm{k} = \langle 0,0,1\rangle$

- Any vector $\mathrm{v} = \langle v_1, v_2, v_3\rangle$ can be written as **a linear combination of standard unit vectors** as follows:

  $$\mathrm{v} = \langle v_1, v_2, v_3\rangle = \langle v_1, 0, 0\rangle + \langle 0, v_2, 0\rangle + \langle 0, 0, v_3\rangle = v_1\mathrm{i} + v_2\mathrm{j} + v_3\mathrm{k}.$$

  Consider a vector $\overrightarrow{PQ}$ with the point $P(x_1, y_1, z_1)$ and point $Q(x_2, y_2, z_2)$. Then $\overrightarrow{PQ}$ can be expressed by

  $$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle = (x_2 - x_1)\mathrm{i} + (y_2 - y_1)\mathrm{j} + (z_2 - z_1)\mathrm{k}$$

Example 3

1. If $\mathrm{u} = \langle 1,-2,3\rangle$ $\mathrm{v} = \langle 0,2,-1\rangle$, express the vector $3\mathrm{u} - 3\mathrm{v}$ in terms of the standard unit(basis) vectors.
2. Find the unit vector in the direction of the vector $\mathrm{i} - 2\mathrm{j} + 2\mathrm{k}$.
3. Find a unit vector in the direction of the vector from $P(1,0,2)$ to $Q(3,2,0)$.