Definition

Definition of Dot Product

The dot product of $\mathrm{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathrm{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathrm{u} \cdot \mathrm{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 .$$

Example1

Find $\mathrm{a} \cdot \mathrm{b}$.

(1) $\mathrm{a} = \langle -2, 1/4 \rangle$, $\mathrm{b} = \langle 1, -4 \rangle$ (2) $\mathrm{a} = \langle -1, -3, 2 \rangle$, $\mathrm{b} = \langle 6, -1/3, 5 \rangle$

Properties of the Dot Product

1. $\mathrm{u} \cdot \mathrm{v} = \mathrm{v} \cdot \mathrm{u}$
2. $(c\mathrm{u}) \cdot \mathrm{v} = \mathrm{u} \cdot (c\mathrm{v}) = c(\mathrm{u} \cdot \mathrm{v})$
3. $\mathrm{u} \cdot (\mathrm{v} + \mathrm{w}) = \mathrm{u} \cdot \mathrm{v} + \mathrm{u} \cdot \mathrm{w}$
4. $\mathrm{u} \cdot \mathrm{u} = |\mathrm{u}|^2$
5. $0 \cdot \mathrm{v} = 0$

Those can be easily proved.

Theorem

If the angle $\theta (0 \le \theta \le \pi)$ *is the angle between two nonzero vectors* $\mathrm{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\mathrm{v} = \langle v_1, v_2, v_3 \rangle$, *then we have*

$$\mathrm{u} \cdot \mathrm{v} = |\mathrm{u}||\mathrm{v}| \cos \theta .$$

- The above theorem can be proved using definition of dot product and laws of cosines. If we apply the theorem, we can find determine the angle between two vectors:

$$\cos \theta = \frac{\mathrm{u} \cdot \mathrm{v}}{|\mathrm{u}||\mathrm{v}|}$$

Example2

Find the angle between $\mathrm{u} = \langle 1, -1, 0 \rangle$ and $\mathrm{v} = \langle 0, 1, 1 \rangle$.

Perpendicular(Orthogonal) Vectors: $\mathrm{u} \perp \mathrm{v} \Leftrightarrow \mathrm{u} \cdot \mathrm{v} = 0$.

Example3

Show that $\mathrm{u} = \langle 2, 1, -3 \rangle$ is perpendicular to $\mathrm{v} = \langle 1, 4, 2 \rangle$.

Vector Projections

- **Scalar component of u in the direction of v(Scalar projection of u onto v):**

$$\text{comp}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$$

- **Vector projection** of u onto v:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

  This means that u is projected onto v.

- **Physical meaning of vector projection:** If u represents a force applied to a box and the box is moving in the direction of v, $\text{proj}_{\mathbf{v}}\mathbf{u}$ represents the effective force in the direction of v.

Example4

Find the vector projection of $\mathbf{u} = 6\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - \mathbf{j} - 3\mathbf{k}$ and the scalar component of u in the direction of v.