12.2 Vectors

- Scalar - A quantity having magnitude(size) but no direction, e.g., mass, length, time, temperature, speed.
- Vector - A quantity having both magnitude and direction, e.g., displacement, velocity, force, acceleration.

1. A vector is represented by a directed line segment. Consider a vector $\overrightarrow{AB}$ with initial point A and terminal point B.
2. $|\overrightarrow{AB}|$: The length (magnitude, size) of the vector $\overrightarrow{AB}$.

Definition

$\overrightarrow{AB} = \overrightarrow{CD}$ if they have **the same length(size) and direction**.

- Several notation of vectors

1. $\overrightarrow{AB}$ with initial point A and terminal point B
2. **a** using lowercase, bold face letter
3. $\overrightarrow{a}$ using lowercase and putting an arrow above the letter

- The zero vector, denoted by **0** has length 0 without directions.

Combining

- **Vector $+$(addition)**: we can **add** two vectors by the **Triangle Law** or the **Parallelogram Law**.
- **Scalar multiplication**: for a scalar c and a vector $\mathbf{u}$ the scalar multiple $c\,\mathbf{u}$ is the vector

1. whose length is $|c|$ times
2. whose direction is the same as $\mathbf{u}$ if $c > 0$
3. whose direction is opposite to $\mathbf{u}$ if $c < 0$

Note that $c\,\mathbf{u} = \mathbf{0}$ if $c = 0$ or $\mathbf{u} = \mathbf{0}$.

- **Vector difference**: we regard the difference of vectors as the sum of vectors, i.e.,

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}),$$

where $-\mathbf{v}$ is called the negative of $\mathbf{v}$.

Components

- If the lowercase letter $\mathbf{u}(\overrightarrow{u})$ is used, the initial point of $\mathbf{u}$ is at the origin and $\mathbf{u}$ is called to be in standard position.
  For example, $\mathbf{u} = \overrightarrow{OP} = \langle u_1, u_2, u_3 \rangle$ is the (standard) position vector of the point $P(u_1, u_2, u_3)$.

Definition

1. The component form of $\mathbf{v}$ is $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, where the initial point is at the origin and terminal point is at (v_1, v_2, v_3).
2. Consider a vector $\overrightarrow{PQ}$ with the point $P(x_1, y_1, z_1)$ and point $Q(x_2, y_2, z_2)$. Then if the standard position vector $\mathbf{v} = \overrightarrow{PQ}$, $\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.
3. The magnitude or length of the vector $\mathbf{v} = \overrightarrow{PQ}$ is

$$
\begin{aligned}
|\mathbf{v}| = |\overrightarrow{PQ}| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\
&= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}
\end{aligned}
$$

Example 1

Find the component form and length of the vector with initial point $P(3,2,1)$ and terminal point $Q(-5,2,2)$.

Definition

Addition(difference) of position vectors and multiplication of a position vector by a scalar.

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with a scalar c.

1. $+$: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$
   $-$: $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$
2. Scalar multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$

- The magnitude of $k\mathbf{u}$ is $|k\mathbf{u}| = |k||\mathbf{u}|$. $-\mathbf{u}$ has the same magnitude as $\mathbf{u}$ but has the opposite direction.

Example 2

Let $\mathbf{u} = \langle 1, -3, 2 \rangle$ $\mathbf{v} = \langle 4, 5, 1 \rangle$. Then find

(1) $2\mathbf{u}$ (2) $3\mathbf{v}$ (3) $2\mathbf{u}+\mathbf{v}$ (4) $2\mathbf{u} - 3\mathbf{v}$ (5) $\left|-\frac{1}{2}\mathbf{u}\right|$

- Properties of vectors

1. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors and a, b be scalars.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{v}+(\mathbf{u}+\mathbf{w})$
4. $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}$ and $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
5. $0\mathbf{u}=\mathbf{0}$ and $1\mathbf{u}=\mathbf{u}$
6. $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$ and $(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$

- A **unit vector** is a vector whose length is 1.
  If $|\mathbf{u}|=1$, $\mathbf{u}$ is a unit vector. For example, $\mathbf{u}=\left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right\rangle$ ...
  Whenever $\mathbf{v} \neq \mathbf{0}$, we have

$$\left| \frac{1}{|\mathbf{v}|}\mathbf{v} \right| = \left| \frac{1}{|\mathbf{v}|} \right| |\mathbf{v}| = \frac{1}{|\mathbf{v}|} |\mathbf{v}| = 1,$$

  which implies that we can change any vector $\mathbf{v}$ into unit vector $\frac{\mathbf{v}}{|\mathbf{v}|}$ in the direction of $\mathbf{v}$. $\frac{\mathbf{v}}{|\mathbf{v}|}$ is called the direction of $\mathbf{v}$.

Definition

Standard unit(basis) vectors: $\mathbf{i} = \langle 1, 0, 0 \rangle$ $\mathbf{j} = \langle 0, 1, 0 \rangle$ $\mathbf{k} = \langle 0, 0, 1 \rangle$

- Any vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be written as **a linear combination of standard unit vectors** as follows:

  $$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

  Consider a vector $\overrightarrow{PQ}$ with the point $P(x_1, y_1, z_1)$ and point $Q(x_2, y_2, z_2)$. Then $\overrightarrow{PQ}$ can be expressed by

  $$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

Example 3

1. If $\mathbf{u} = \langle 1, -2, 3 \rangle$ $\mathbf{v} = \langle 0, 2, -1 \rangle$, express the vector $3\mathbf{u} - 3\mathbf{v}$ in terms of the standard unit(basis) vectors.
2. Find the unit vector in the direction of the vector $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.
3. Find a unit vector in the direction of the vector from $P(1, 0, 2)$ to $Q(3, 2, 0)$.