Definition

Definition of Dot Product

The dot product of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Example1

Find $\mathbf{a} \cdot \mathbf{b}$.

(1) $\mathbf{a} = \langle -2, 1/4 \rangle$, $\mathbf{b} = \langle 1, -4 \rangle$ (2) $\mathbf{a} = \langle -1, -3, 2 \rangle$, $\mathbf{b} = \langle 6, -1/3, 5 \rangle$

Properties of the Dot Product

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{v} = 0$

Those can be easily proved.

Theorem

If the angle $\theta(0 \le \theta \le \pi)$ *is the angle between two nonzero vectors* $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, *then we have*

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta.$$

- The above theorem can be proved using definition of dot product and laws of cosines. If we apply the theorem, we can find determine the angle between two vectors:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

Example2

Find the angle between $\mathbf{u} = \langle 1, -1, 0 \rangle$ and $\mathbf{v} = \langle 0, 1, 1 \rangle$.

Perpendicular(Orthogonal) Vectors: $\mathbf{u} \perp \mathbf{v} \Leftrightarrow \mathbf{u} \cdot \mathbf{v} = 0$.

Example3

Show that $\mathbf{u} = \langle 2, 1, -3 \rangle$ is perpendicular to $\mathbf{v} = \langle 1, 4, 2 \rangle$.

Vector Projections

- **Scalar component of u in the direction of v(Scalar projection of u onto v):**

$$\text{comp}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}|\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$$

- **Vector projection** of $\mathbf{u}$ onto $\mathbf{v}$:

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

  This means that $\mathbf{u}$ is projected onto $\mathbf{v}$.

- **Physical meaning of vector projection:** If $\mathbf{u}$ represents a force applied to a box and the box is moving in the direction of $\mathbf{v}$, $\text{proj}_{\mathbf{v}}\mathbf{u}$ represents the effective force in the direction of $\mathbf{v}$.

Example4

Find the vector projection of $\mathbf{u} = 6\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - \mathbf{j} - 3\mathbf{k}$ and the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$.