- The cross product is defined in the 3-dimensional system and is used to describe how a plane is tilting.
- The cross product has too many applications of engineering and physics.

Fact

1. *How to find the* ***determinant of*** 3×3 ***matrix****?*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

2. *How to find the* ***determinant of*** 2×2 ***matrix****?*

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

- For $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, we can express $\mathbf{u} \times \mathbf{v}$ in the expansion of the symbolic determinant:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}.$$

Example 1

Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

Definition

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin\theta)\,\mathbf{n}.$$

- If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$ are not parallel, they determine a plane. Then we select a unit vector $\mathbf{n}$ perpendicular to the plane by the right-hand rule, which implies that the vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

Theorem

The vector $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}$ *and* $\mathbf{v}$.

- The Theorem can be proved by using $\mathbf{a} \cdot \mathbf{b} = 0 \Leftrightarrow \mathbf{a} \perp \mathbf{b}$.

Corollary

Parallel Vectors

$$\mathbf{u} // \mathbf{v} \Leftrightarrow \mathbf{u} \times \mathbf{v} = \mathbf{0}.$$

Fact

$|\mathbf{u} \times \mathbf{v}|$ *is the area of a parallelogram.*

- It follows from the definition of the cross product that $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}||\sin\theta||\mathbf{n}| = |\mathbf{u}||\mathbf{v}|\sin\theta$

Example2

1. Find a vector perpendicular to the plane of $P(-1,1,0)$, $Q(2,-1,1)$, and $R(-1,1,2)$. Note that the vector is not unique.
2. Find the area of the triangle with vertices P, Q, and R.

- For the standard unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ we can see $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$ and $\mathbf{i} \times \mathbf{i} = \mathbf{0}, \quad \mathbf{j} \times \mathbf{j} = \mathbf{0}, \quad \mathbf{k} \times \mathbf{k} = \mathbf{0}$.
- **Properties of the Cross Product**

1. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
2. $(r\mathbf{u}) \times \mathbf{v} = r(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (r\mathbf{v})$
3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
5. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{u} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$
7. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$

- Those properties can be shown easily by writing the component form of the vectors and using the symbolic expression.
- Note that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

Definition

Triple Scalar or Box Product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$
$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ is the volume of the parallelepiped, since
$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\mathbf{u}||\mathbf{v} \times \mathbf{w}||\cos\theta|$.

- The triple scalar product can be calculated as the determinant:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

- By using the determinant, we can see that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$.

Example3

Find the volume of the box(parallelepiped) determined by
$\mathbf{u} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + \mathbf{k}$ and $\mathbf{w} = 5\mathbf{j} - 4\mathbf{k}$.