13.2 Derivatives and Integrals of Vector functions

Definition

Suppose that the vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, h are differentiable at t. Then the **Derivative of the Vector function $\mathbf{r}(t)$** is

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k} = \left\langle \frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt} \right\rangle.$$

- The vector $\mathbf{r}'(a)$ is called the **tangent vector** at the point $P(f(a), g(a), h(a))$ to the curve C defined by $\mathbf{r}$, provided that $\mathbf{r}'(a)$ exists and $\mathbf{r}'(a) \neq \mathbf{0}$. The **Unit Tangent Vector** is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

- The **tangent line** to the curve C at P is defined to be the line through P parallel to the tangent line vector $\mathbf{r}'(a)$.
- We need to figure out its geometric significance.

Example1

1. Find the derivatives of $\mathbf{r}(t) = \langle 2 + t^2, te^{2t}, \cos 2t \rangle$.
2. Find the unit tangent vector at the given time $t = 0$.

Example2

1. For the curve $\mathbf{r}(t) = \langle \sqrt{t}, 1 - t \rangle$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and tangent vector $\mathbf{r}'(1)$.
2. Find parametric equations for the tangent line to the helix with parametric equations

$$x = \cos t \quad y = \sin t \quad z = t$$

at the point $(0, 1, \pi/2)$.

- **Differentiation Rules for vector functions**

  Let $\mathbf{u}, \mathbf{v}$ be smooth vector function of t and $\mathbf{C}$ be a constant vector and c be any scalar and f any smooth scalar function.

1. Constant Function Rule: $\frac{d\mathbf{C}}{dt} = \mathbf{0}$.
2. Scalar Multiple Rules: $\frac{d}{dt}(c\mathbf{u}) = c\frac{d\mathbf{u}}{dt}$ and $\frac{d}{dt}(f\mathbf{u}) = \frac{df}{dt}\mathbf{u} + f\frac{d\mathbf{u}}{dt}$.
3. Sum and Difference Rule: $\frac{d}{dt}(\mathbf{u} \pm \mathbf{v}) = \frac{d\mathbf{u}}{dt} \pm \frac{d\mathbf{v}}{dt}$.
4. Dot product rule: $\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \frac{d\mathbf{u}}{dt} \cdot \mathbf{v} + \mathbf{u} \cdot \frac{d\mathbf{v}}{dt}$
5. Cross product Rule: $\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \frac{d\mathbf{u}}{dt} \times \mathbf{v} + \mathbf{u} \times \frac{d\mathbf{v}}{dt}$.
6. Chain Rule: $\frac{d}{dt}[\mathbf{u}(f(t))] = \frac{df}{dt}\frac{d\mathbf{u}}{dt}$.

- **Vector function of Constant Length.**

1. $|\mathbf{u}(t)| = c$ for any time $t \Rightarrow \mathbf{u}'(t) \perp \mathbf{u}$, i.e., $\mathbf{u}'(t) \cdot \mathbf{u}(t) = 0$.
2. Geometrically this implies that $\mathbf{u}'(t) \perp \mathbf{u}(t)$ at any points on a circle (or sphere) for any time t.

Example3

Show that $\mathbf{r}(t) = \langle \sin t, \cos t, 1 \rangle$ has a constant length and is orthogonal to its derivative.

Definition

Indefinite Integral for the Vector Function $\mathbf{u}(t) = \langle f(t), g(t), h(t) \rangle$:

$$\int \mathbf{u}(t)dt = \left\langle \int f(t)dt, \int g(t)dt, \int h(t)dt \right\rangle.$$

Example4

Find the following indefinite integral

$$\int \langle \cos t, \sin t, t \rangle \, dt.$$

Definition

Definite Integral for the Vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$:
The definite integral of $\mathbf{r}$ from a to b is

$$\begin{aligned} \int_a^b \mathbf{r}(t)dt &= \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k} \\ &= \left\langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right\rangle. \end{aligned}$$

- Note that f, g, h are integrable over $[a, b] \Leftrightarrow$ Vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is integrable over $[a, b]$.

Example5

Evaluate the following definite Integral

$$\int_0^{\pi/2} ((\cos t)\mathbf{i} + (\sin t)\mathbf{j} - 2t\mathbf{k})dt.$$