Fact

Arc Length formula

The Length of a smooth curve $\mathbf{u}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \leq t \leq b$ *is*

$$L = \int_a^b |\mathbf{v}|dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt,$$

where $\mathbf{v} = (dx/dt)\mathbf{i} + (dy/dt)\mathbf{j} + (dz/dt)\mathbf{k}$.

Example1:

A glider is soaring upward along the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$. How far does the glider travel along its path from $t = 0$ to $t = 4\pi$?

Definition

Arc Length Parameter with Base Point $P(t_0)$

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)| d\tau.$$

- Applying the Fundamental Theorem Part I, from the previous definition we can derive

$$\frac{ds}{dt} = |\mathbf{v}(t)|.$$

Example2:

Find the arc length parameter (reparametrize a curve w.r.t. arc length) along the helix

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

measured from $(1, 0, 0)$ in the direction of increasing t.

- **Unit Tangent Vector T**: Since the velocity vector $\mathbf{v} = d\mathbf{u}/dt$ is tangent to the curve, the unit vector $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is a unit vector tangent to the smooth curve.

Example3:

Find the unit tangent vector of the curve

$$\mathbf{r}(t) = (2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + t^2\mathbf{k}.$$

- **Curvature:** the rate at which $\mathbf{T}$ turns **per unit of length** along the curve.

Definition

If $\mathbf{T}$ is the unit tangent vector of a smooth curve, the **curvature** κ is defined as

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|.$$

1. If $|d\mathbf{T}/ds|$ is large, $\mathbf{T}$ turns sharply.
2. If $|d\mathbf{T}/ds|$ is small, $\mathbf{T}$ turns more slowly.

- In a practical way, we use the following formula: if $\mathbf{u}(t)$ is a smooth curve, then curvature is

$$\kappa(t) = \frac{1}{\left|\frac{d\mathbf{u}}{dt}\right|}\left|\frac{d\mathbf{T}(t)}{dt}\right|.$$

Example4:

The curvature of a straight line is zero, since on a straight line the unit tangent vector $\mathbf{T}$ has the same direction.

Example5:

Prove that the curvature of a circle of radius a is $1/a$.

Theorem

For the vector valued function $\mathbf{u}$,

$$\kappa = \frac{|\mathbf{u}'(t) \times \mathbf{u}''(t)|}{|\mathbf{u}'(t)|^3}.$$

Example6

Find the curvature of $\mathbf{u}(t) = \langle t, t^2, t^3 \rangle$ for any $t \geq 0$ and at the point $(1,1,1)$.

- We can apply the previous Theorem to the scalar function $y = f(x)$. Letting $\mathbf{u}(x) = \langle x, f(x), 0 \rangle$ we can have

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

Example7

Find the curvature of the paprabola $f(x) = x^2$ at $x = 0, -1, -2$.

1. **The Principal unit Normal Vectors:**

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

2. **The Binormal Vectors:**

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

3. **The Normal plane:** determined by the vectors **N** and **B**.
4. **The Osculating plane:** determined by the vectors **T** and **N**.
5. **The Osculating circle:** better approximation at certain points than the linear approximation

Example8:

find the equations of the normal plane and osculating plane of the helix at point $P(1,0,0)$:

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}.$$

- Circle of Curvature for curves on a plane: for $\kappa \neq 0$ circle of Curvature (osculating circle) at a point P on a plane is the circle in the plane of the curve that

1. has the same tangent line as the curve does
2. has the same curvature as the curve does at P.
3. lies toward the concave or inner side of the curve.

- The radius of curvature of the curve at P is the radius of the circle of the curvature:

$$\text{Radius of curvature of the curve} = \frac{1}{\kappa}.$$

Example 9:

1. Find and graph the osculating circle of the parabola $y = x^2$ at the origin.
2. Find and graph the osculating circle of the parabola $y = -x^2$ at $x = -1/2$.