- For functions with a single variable:
  Let $u = f(x)$ and $x = g(t)$. If there exit f' and g', then

$$f(g(t))' = f'(g(t))g'(t)$$
$$\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} \text{ in Leibniz's notation}$$

Example1

1. Find the derivative of $u = \left(x^3 + 1\right)^{10}$.
2. Find the slope of tangent line to the curve $u = \cos^2 t$ at $t = \pi/4$.

- However, for functions with more than two variables the Chain rule has **several forms**.

Theorem

Case I: *If $u = f(x, y)$ has continuous f_x and f_y and there exist $x'(t)$ and $y'(t)$, then the composite $u = f(x(t), y(t))$ is differentiable function of t and*

$$\frac{df}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t) \; or$$
$$\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Example1:

1. Apply the Chain rule to find the derivative of

$$u = x\,y$$

with respect to t along the path $x(t) = \cos t$, $y(t) = \sin t$. What is the derivative's value at $t = \pi/4$?

2. If $z = xy^2 + 3x^2y$, where $x(t) = \cos t$ and $y(t) = \sin 2t$, find dz/dt when $t = 0$.

Theorem

Case II *Suppose that $u = f(x, y)$, $x = g(s, t)$, $y = h(s, t)$ are differentiable, then u has partial derivatives w.r.t. r and s given by*

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds},\\ \frac{\partial u}{\partial t} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\end{aligned}$$

- s, t are independent variables and x, y are called intermediate variables and u is a dependent variable.

Example2

1. Express $\partial u/\partial t$ and $\partial u/\partial s$ in terms of t and s if

$$u = 2x + y, \quad x = \frac{t}{s}, \quad y = t^2 + \ln s.$$

2. Express $\partial u/\partial t$ and $\partial u/\partial s$ in terms of t and s if $u(s, t) = e^r \sin\theta,\ r(s, t) = s\,t,\ \theta(s, t) = \sqrt{s^2 + t^2}$..

Theorem

General Version *Suppose that* $u = f(x_1, x_2, \cdots, x_n)$ *is a differentiable function of the variables* $x_1, x_2, \cdots, x_n$ *and* $x_1, x_2, \cdots, x_n$ *are differentiable functions of the variable of* $t_1, t_2, \cdots, t_m$. *Then* u *is a differentiable function of the variables* $t_1, t_2, \cdots, t_m$ *and the partial derivatives of* u *are given by the form*

$$\begin{aligned}
\frac{\partial u}{\partial t_1} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_1}, \\
\frac{\partial u}{\partial t_2} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_2}, \\
\cdots & \quad \cdots \\
\frac{\partial u}{\partial t_m} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_m}.
\end{aligned}$$

- It is helpful to draw the tree diagram in order to memorize the chain rules.

Example3

Use a tree diagram to write out the Chain Rule for the given case.
1. $u = f(x, y)$, where $x = x(r, s, t)$, $y = y(r, s, t)$.
2. $w = f(x, y, z, t)$, where $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, and $t = t(u, v)$.

Example4

1. If $R = \ln\left(u^2 + v^2\right)$, where $u = 2x + y$ and $v = x - 3y$, find the value of $\partial R/\partial x$ and $\partial R/\partial y$, when $x = y = 1$.
2. If $u = x^3 y + y^2 z^4$, where $x = r s e^{-t}$, $y = r^2 s \ln(t+1)$, $z = r s^2 \sin t$, find the value of $\partial u/\partial s$, when $r = 1$, $s = -1$, $t = 0$.

- **Implicit Differentiation**

1. Suppose that $F(x, y)$ is differentiable and that $F(x, y(x)) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,
$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$
2. Suppose that $F(x, y, z)$ is differentiable and that $F(x, y, z(x, y)) = 0$ defines z as a differentiable function of x and y. Then at any point where $F_z \neq 0$,
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

- However, I do **not** suggest to memorize those formulas.

Example5

1. Find dy/dx if $y^2 - x^2 - \sin xy = 0$, where $y = y(x)$.
2. Find $\partial z/\partial x$ and $\partial z/\partial y$ if $x^3 + y^3 + z^3 + 2xyz = 5$. Also find $\partial z/\partial x|_{(1,1,1)}$ and $\partial z/\partial y|_{(1,1,1)}$. Note that $z = z(x, y)$.