- Directional Derivatives in the plane
  Our main interest is the rate of change of a function $f(x(t), y(t))$ along a straight line.
  Recall a line equation through a point $P_0\,(x_0, y_0)$ and parallel to the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$. Then the line equation is

$$x(t) = x_0 + t\,u_1 \text{ and } y(t) = y_0 + t\,u_2.$$

Definition

Directional Derivative: The derivative of $f(x, y)$ at $P_0\,(x_0, y_0)$ in the direction of the **unit vector** $\mathbf{u} = \langle u_1, u_2 \rangle$ is the scalar

$$D_{\mathbf{u}} f\,(x_0, y_0) = \lim_{t \to 0} \frac{f\,(x_0 + tu_1, y_0 + tu_2) - f\,(x_0, y_0)}{t},$$

provided the limit exists.

- Other Notations for the directional derivative at a point $P_0\left(x_0, y_0\right)$ are

$$D_{\mathrm{u}} f\left(x_0, y_0\right)=\left(D_{\mathrm{u}} f\right)_{P_0}=\left(\frac{df}{dt}\right)_{\mathrm{u}, P_0}.$$

- Note that a physical interpretation $\left(D_{\mathrm{u}} T\right)$ is the instantaneous rate of change of temperature in the direction u.
- If the unit vector $\mathrm{u}=\left\langle u_1, u_2\right\rangle$ or $\mathrm{u}=\langle\cos \theta, \sin \theta\rangle$, then

$$D_{\mathrm{u}} f(x, y)=f_x(x, y) u_1+f_y(x, y) u_2=f_x(x, y) \cos \theta+f_y(x, y) \sin \theta,$$

where θ is an angle with the positive $x-$axis.

Example1

Find the directional derivatives $D_{\mathrm{u}} f(x, y)$ if

$$f(x, y)=x^3-2 x y+3 y^2$$

and u is the unit vector given by angle $\theta=\pi / 3$. What is $D_{\mathrm{u}} f(1,2)$?

- Gradient is derived from the Directional Derivatives

1. The rate of change of $f(x(t), y(t))$ with respect to t is

$$D_{\mathrm{u}} f(x,y) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt},$$

where $x(t)$ and $y(t)$ are differentiable curve.

2. Then the rate of change of $f(x,y)$ at $P_0\,(x_0, y_0)$ in the direction of the **unit vector** $\mathrm{u} = \langle u_1, u_2 \rangle$ along the line $x = x_0 + tu_1$ and $y = y_0 + tu_2$ is

$$\begin{aligned} D_{\mathrm{u}} f\,(x_0, y_0) &= \left(\frac{\partial f}{\partial x}\right)_{P_0} \frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \frac{dy}{dt} \\ &= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2 \\ &= \left\langle \left(\frac{\partial f}{\partial x}\right)_{P_0}, \left(\frac{\partial f}{\partial y}\right)_{P_0} \right\rangle \cdot \langle u_1, u_2 \rangle . \end{aligned}$$

Definition

Gradient vector: The gradient of $f(x,y)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle.$$

- The directional derivatives can be written as

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u},$$

which expresses the directional derivative in the direction of $\mathbf{u}$ as the scalar projection of the gradient vector onto $\mathbf{u}$.

Example2:

1. Find the directional derivative of the function $f(x,y) = x^3y^2 - 2y$ at the point $(-2,1)$ in the direction of the vector $\mathbf{v} = \langle 2, -3 \rangle$.
2. Find the directional derivative of $f(x,y) = ye^x + \sin(xy)$ at the point $(1,0)$ in the direction of $\mathbf{v} = \mathbf{i} - \sqrt{3}\mathbf{j}$.

- Functions of Three Variables

Definition

The derivative of $f(x, y)$ at $P_0(x_0, y_0, z_0)$ in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is the scalar

$$\left(\frac{df}{dt}\right)_{\mathbf{u},P_0} = \lim_{t\to 0} \frac{f(x_0 + tu_1, y_0 + tu_2, z_0 + tu_3) - f(x_0, y_0, z_0)}{t},$$

provided the limit exists.

Definition

Gradient vector: The gradient of $f(x, y, z)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

- The dirctional derivatives can be written as

$$D_{\mathrm{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \mathrm{u}$$

Example3 If $f(x,y,z) = x\cos(yz)$,

1. Find the gradient of f
2. Find the directional derivative of f at $(-1,0,0)$ in the direction of $\mathrm{u} = \langle 2,3,-1 \rangle$.

- Properties of the Directional Derivative
  $D_{\mathrm{u}} f = \nabla f \cdot \mathrm{u} = |\nabla f||\mathrm{u}|\cos\theta = |\nabla f|\cos\theta$

1. The function f increases most rapidly when u is the direction of ∇f, i.e., $\theta = 0$.
2. The function f decreases most rapidly when u is the direction of $-\nabla f$, i.e., $\theta = \pi$.
3. Any direction u orthogonal to a $\nabla f \neq 0$ is a direction of zero change in f.

Example 4:

1. If $f(x,y) = x^2 + y^2$, find the rate of change of f at the point $P(1,1)$ in the direction from P to $Q(2,3)$.
2. In what direction does f increase most rapidly at $(1,1)$?
3. In what direction does f decrease most rapidly at $(1,1)$?
4. What are the directions of zero change in f at $(1,1)$?

- Tangent Planes and Normal Lines

  Consider the smooth curve $\mathrm{r}(t) = g(t)\mathrm{i} + h(t)\mathrm{j} + k(t)\mathrm{k}$ on the level surface $f(x,y,z) = c$. $\Rightarrow f((g(t), h(t), k(t)) = c$. Using the Chain rule to Differentiate both sides, we have

  $$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dg}{dt}, \frac{dh}{dt}, \frac{dk}{dt} \right\rangle = 0$$

  So ∇f is normal to the curve's velocity vector $d\mathrm{r}/dt$ on the level surface.

- Tangent Plane and Normal Line

  The tangent plane to the **level surface** $f(x,y,z)=c$ at the point $P_0\,(x_0,y_0,z_0)$:

$$\left.\frac{\partial f}{\partial x}\right|_{P_0}(x-x_0)+\left.\frac{\partial f}{\partial y}\right|_{P_0}(y-y_0)+\left.\frac{\partial f}{\partial z}\right|_{P_0}(z-z_0)=0$$

  The Normal line to the **level surface** $f(x,y,z)=c$ at the point $P_0\,(x_0,y_0,z_0)$:

$$x=x_0+f_x\,(P_0)\,t,\; y=y_0+f_y\,(P_0)\,t,\; z=z_0+f_z\,(P_0)\,t$$

Example 5:

Find the tangent plane and normal line of the surface $f(x,y,z)=x^2+y^2+z-4=0$ at $P_0(1,1,2)$.

- Let $F(x,y,z)=f(x,y)-z=0$. The plane tangent to a surface $z=f(x,y)$ at $P_0\,(x_0,y_0,z_0)$:

$$f_x\,(x_0,y_0)\,(x-x_0)+f_y\,(x_0,y_0)\,(y-y_0)-(z-z_0)=0.$$