- The definition of Limit of a function with several variables is similar to the definition of Limit with a single variable. However, there is a crucial difference between them: if (x_0, y_0) lies in the interior of function f's domain, (x, y) can approach (x_0, y_0) from **any direction**.

Definition

Limit of a function of two variables

If for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \Rightarrow |f(x, y) - L| < \varepsilon,$$

then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L.$$

Theorem

Properties of Limits of Functions of Two Variables: Suppose that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \textit{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

Constant Multiple Rule: $\lim_{(x,y)\to(x_0,y_0)} k\,f(x,y) = k\,L$

Sum and Difference of Rule:

$\lim_{(x,y)\to(x_0,y_0)}(f(x,y) \pm g(x,y)) = L \pm M$

Product Rule: $\lim_{(x,y)\to(x_0,y_0)}(f(x,y)\cdot g(x,y)) = L\cdot M$

Quotient Rule: $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$

Power Rule: $\lim_{(x,y)\to(x_0,y_0)}(f(x,y))^{r/s} = L^{r/s}$.

Example1:

Find each of following limits

1.
$$\lim_{(x,y)\to(0,1)} \frac{x-2xy+2}{x^2y+3xy-y^3}$$

2.
$$\lim_{(x,y)\to(1,-2)} \sqrt{x^2+y^2}$$

3.
$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}.$$

Fact

Two-Path Test for Nonexistence of a Limit

If a function $f(x,y)$ has different limits along two different paths as $(x,y)\to(x_0,y_0)$, then

$$\nexists \lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

Theorem

Sandwich(Squeeze) Theorem

Suppose that $g(x,y) \le f(x,y) \le h(x,y)$ *in a neighborhood of* (x_0, y_0) *and* $\lim_{(x,y)\to(x_0,y_0)} g(x,y) = \lim_{(x,y)\to(x_0,y_0)} h(x,y) = L$. *Then we have* $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$.

Example2:

1. Show that

$$\nexists \lim_{(x,y)\to(0,0)} \frac{2x^2 - y^2}{x^2 + y^2}$$

2. Using the Sandwich Theorem, find

$$\lim_{(x,y)\to(0,0)} \frac{2x^2 y}{x^2 + y^2} \text{ if it exists.}$$

3. Show that

$$\nexists \lim_{(x,y)\to(0.0)} f(x,y) = \frac{x^2 y}{x^4 + y^2}$$

Definition

A function $f(x,y)$ is **continuous** at the point (x_0,y_0) if

1. f is defined at (x_0,y_0)
2. $$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \text{ exists}$$
3. $$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

- Note that a function is continuous if it is continuous at every point of its domain.

Example3:

Show that

$$f(x,y)=\begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$$

is not continuous at the origin.