Definition

1. Partial Derivative with respect to x at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

2. Partial Derivative with respect to y at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h},$$

provided the limit exists.

- **Notation** for a partial derivative of $z = f(x, y)$.

1. For the points (x_0, y_0): $\frac{\partial f}{\partial x}(x_0, y_0)$ or $\frac{\partial z}{\partial x}(x_0, y_0)$ or $f_x(x_0, y_0)$
2. For a function: $\frac{\partial f}{\partial x}$, f_x, $\frac{\partial z}{\partial x}$, z_x

- Rule for Finding Partial Derivatives of $z = f(x,y)$

1. To find f_x, regard y as a constant and differentiate $f(x,y)$ w.r.t. x.
2. To find f_y, regard x as a constant and differentiate $f(x,y)$ w.r.t. y.

Example1

1. If $f(x,y) = 2x^3 + x^3y^2 - y^2$, find $f_x(1,2)$ and $f_y(1,1)$.
2. Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point at $(2,-1)$ if $f(x,y) = x^2 + 2xy + 3y + 1$.
3. Find $\partial f/\partial x$ if $f(x,y) = x\cos xy$.
4. Find f_x and f_y if

$$f(x,y) = \frac{2x}{x + \sin y}.$$

5. If

$$f(x,y) = \sin\left(\frac{y}{x+y}\right),$$

calculate f_x and f_y.

- Geometric iterpretation of Partial Derivatives:

1. $\partial f/\partial x$ at (x_0, y_0) gives the rate of change of f with respect to x with fixed y at y_0 which is the rate of f in the direction of $\mathbf{i}$ at (x_0, y_0).
2. $\partial f/\partial y$ at (x_0, y_0) is similar to the definition of the partial Derivative with respect to x at the point (x_0, y_0).

Example2

If $f(x,y) = 1 - x^2 - 2y^2$, find $f_x(1,1)$ and $f_y(1,1)$ and interpret these numbers as slopes.

Example3

1. Find z_x and z_y if z is defined implicitly as a function of x and y by the equation

$$x^2 + y^3 + z^2 + 3xyz = 2.$$

2. Find f_x, f_y and f_z if $f(x,y,z) = e^x \ln yz$

- The second-order Partial Derivatives and their Notations:

$\frac{\partial^2 f}{\partial x^2}$ or f_{xx}

$\frac{\partial^2 f}{\partial y^2}$ or f_{yy}

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ or f_{yx}

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ or f_{xy}

Example4

If $f(x, y) = x \sin y + ye^{-x}$, find

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and } \frac{\partial^2 f}{\partial x \partial y}$$

Theorem

Clairaut's Theorem (The Mixed Derivative Theorem)
If $f(x,y)$ and its partial derivatives f_x, f_y, and f_{yx} are defined throughout an open disk D containing a point (a,b) and are all continuous at (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Example5

Verify $u_{xy} = u_{yx}$ if $u(x,y) = x\sin(2x+y)$.

- Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Solutions $u(x,y)$ of the equation are called harmonic functions.

Example6

Show that the function $u(x,y) = e^x \cos y$ is a harmonic function.