# 14.5 The Chain Rule

- For functions with a single variable:
  Let $u = f(x)$ and $x = g(t)$. If there exit $f'$ and $g'$, then

$$f(g(t))' = f'(g(t))g'(t)$$
$$\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} \text{ in Leibniz's notation}$$

**Example1**

1. Find the derivative of $u = \left(x^3 + 1\right)^{10}$.
2. Find the slope of tangent line to the curve $u = \cos^2 t$ at $t = \pi/4$.

- However, for functions with more than two variables the Chain rule has **several forms**.

## Theorem

***Case I**: If $u = f(x,y)$ has continuous $f_x$ and $f_y$ and there exist $x'(t)$ and $y'(t)$, then the composite $u = f(x(t),y(t))$ is differentiable function of $t$ and*

$$\frac{df}{dt} = f_x(x(t),y(t)) \cdot x'(t) + f_y(x(t),y(t)) \cdot y'(t) \text{ or}$$
$$\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

## Example1:

1. Apply the Chain rule to find the derivative of

$$u = x\,y$$

with respect to $t$ along the path $x(t) = \cos t$, $y(t) = \sin t$. What is the derivative's value at $t = \pi/4$?
2. If $z = xy^2 + 3x^2y$, where $x(t) = \cos t$ and $y(t) = \sin 2t$, find $dz/dt$ when $t = 0$.

## Theorem

**Case II** *Suppose that $u = f(x, y)$, $x = g(s, t)$, $y = h(s, t)$ are differentiable, then $u$ has partial derivatives w.r.t. $r$ and $s$ given by*

$$\begin{aligned} \frac{\partial u}{\partial s} &= \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}, \\ \frac{\partial u}{\partial t} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \end{aligned}$$

- $s$, $t$ are independent variables and $x$, $y$ are called intermediate variables and $u$ is a dependent variable.

## Example2

1. Express $\partial u/\partial t$ and $\partial u/\partial s$ in terms of $t$ and $s$ if

$$u = 2x + y, \quad x = \frac{t}{s}, \quad y = t^2 + \ln s.$$

2. Express $\partial u/\partial t$ and $\partial u/\partial s$ in terms of $t$ and $s$ if $u(s, t) = e^r \sin\theta$, $r(s, t) = s\,t$, $\theta(s, t) = \sqrt{s^2 + t^2}$..

## Theorem

***General Version*** *Suppose that* $u = f(x_1, x_2, \cdots, x_n)$ *is a differentiable function of the variables* $x_1, x_2, \cdots, x_n$ *and* $x_1, x_2, \cdots, x_n$ *are differentiable functions of the variable of* $t_1, t_2, \cdots, t_m$. *Then* $u$ *is a differentiable function of the variables* $t_1, t_2, \cdots, t_m$ *and the partial derivatives of* $u$ *are given by the form*

$$\begin{aligned}
\frac{\partial u}{\partial t_1} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_1}, \\
\frac{\partial u}{\partial t_2} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_2}, \\
\cdots & \quad \cdots \\
\frac{\partial u}{\partial t_m} &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_m}.
\end{aligned}$$

- It is helpful to draw the tree diagram in order to memorize the chain rules.

### Example3

Use a tree diagram to write out the Chain Rule for the given case.
1. $u = f(x, y)$, where $x = x(r, s, t)$, $y = y(r, s, t)$.
2. $w = f(x, y, z, t)$, where $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$, and $t = t(u, v)$.

### Example4

1. If $R = \ln\left(u^2 + v^2\right)$, where $u = 2x + y$ and $v = x - 3y$, find the value of $\partial R/\partial x$ and $\partial R/\partial y$, when $x = y = 1$.
2. If $u = x^3 y + y^2 z^4$, where $x = r\,s\,e^{-t}$, $y = r^2 s \ln(t + 1)$, $z = r\,s^2 \sin t$, find the value of $\partial u/\partial s$, when $r = 1$, $s = -1$, $t = 0$.

- **Implicit Differentiation**

1. Suppose that $F(x,y)$ is differentiable and that $F(x,y(x))=0$ defines $y$ as a differentiable function of $x$. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

2. Suppose that $F(x,y,z)$ is differentiable and that $F(x,y,z(x,y))=0$ defines $z$ as a differentiable function of $x$ and $y$. Then at any point where $F_z \neq 0$,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}.$$

- However, I do **not** suggest to memorize those formulas.

**Example5**

1. Find $dy/dx$ if $y^2 - x^2 - \sin xy = 0$, where $y = y(x)$.
2. Find $\partial z/\partial x$ and $\partial z/\partial y$ if $x^3 + y^3 + z^3 + 2xyz = 5$. Also find $\partial z/\partial x|_{(1,1,1)}$ and $\partial z/\partial y|_{(1,1,1)}$. Note that $z = z(x,y)$.