- Directional Derivatives in the plane
  Our main interest is the rate of change of a function $f(x(t), y(t))$ along a straight line.
  Recall a line equation through a point $P_0(x_0, y_0)$ and parallel to the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$. Then the line equation is

$$x(t) = x_0 + t\,u_1 \text{ and } y(t) = y_0 + t\,u_2.$$

Definition

Directional Derivative: The derivative of $f(x, y)$ at $P_0(x_0, y_0)$ in the direction of the **unit vector** $\mathbf{u} = \langle u_1, u_2 \rangle$ is the scalar

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t},$$

provided the limit exists.

- Other Notations for the directional derivative at a point $P_0(x_0, y_0)$ are

$$D_{\mathbf{u}} f(x_0, y_0) = (D_{\mathbf{u}} f)_{P_0} = \left(\frac{df}{dt}\right)_{\mathbf{u}, P_0}.$$

- Note that a physical interpretation $(D_{\mathbf{u}} T)$ is the instantaneous rate of change of temperature in the direction $\mathbf{u}$.
- If the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ or $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$, then

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) u_1 + f_y(x, y) u_2 = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta,$$

where θ is an angle with the positive $x-$axis.

Example1

Find the directional derivatives $D_{\mathbf{u}} f(x, y)$ if

$$f(x, y) = x^3 - 2xy + 3y^2$$

and $\mathbf{u}$ is the unit vector given by angle $\theta = \pi/3$. What is $D_{\mathbf{u}} f(1, 2)$?

- Gradient is derived from the Directional Derivatives

1. The rate of change of $f(x(t), y(t))$ with respect to t is

$$D_{\mathbf{u}} f(x,y) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt},$$

where $x(t)$ and $y(t)$ are differentiable curve.

2. Then the rate of change of $f(x,y)$ at $P_0\,(x_0, y_0)$ in the direction of the **unit vector** $\mathbf{u} = \langle u_1\,,\, u_2\rangle$ along the line $x = x_0 + tu_1$ and $y = y_0 + tu_2$ is

$$\begin{aligned} D_{\mathbf{u}} f\,(x_0, y_0) &= \left(\frac{\partial f}{\partial x}\right)_{P_0} \frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \frac{dy}{dt} \\ &= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2 \\ &= \left\langle \left(\frac{\partial f}{\partial x}\right)_{P_0}, \left(\frac{\partial f}{\partial y}\right)_{P_0} \right\rangle \cdot \langle u_1\,,\, u_2\rangle\,. \end{aligned}$$

Definition

Gradient vector: The gradient of $f(x,y)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle.$$

- The directional derivatives can be written as

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u},$$

which expresses the directional derivative in the direction of $\mathbf{u}$ as the scalar projection of the gradient vector onto $\mathbf{u}$.

Example2:

1. Find the directional derivative of the function $f(x,y) = x^3y^2 - 2y$ at the point $(-2,1)$ in the direction of the vector $\mathbf{v} = \langle 2, -3 \rangle$.
2. Find the directional derivative of $f(x,y) = ye^x + \sin(xy)$ at the point $(1,0)$ in the direction of $\mathbf{v} = \mathbf{i} - \sqrt{3}\mathbf{j}$.

- Functions of Three Variables

Definition

The derivative of $f(x,y)$ at $P_0(x_0,y_0,z_0)$ in the direction of the unit vector $\mathbf{u}=\langle u_1,\, u_2,\, u_3\rangle$ is the scalar

$$\left(\frac{df}{dt}\right)_{\mathbf{u},P_0}=\lim_{t\to 0}\frac{f\left(x_0+tu_1,y_0+tu_2,z_0+tu_3\right)-f\left(x_0,y_0,z_0\right)}{t},$$

provided the limit exists.

Definition

Gradient vector: The gradient of $f(x,y,z)$ is the vector

$$\nabla f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}=\left\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle.$$

- The dirctional derivatives can be written as

$$D_{\mathbf{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \mathbf{u}$$

Example3

If $f(x,y,z) = x\cos(yz)$,
1. Find the gradient of f
2. Find the directional derivative of f at $(-1,0,0)$ in the direction of $\mathbf{u} = \langle 2,3,-1 \rangle$.

- Properties of the Directional Derivative
$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$

1. The function f increases most rapidly when $\mathbf{u}$ is the direction of ∇f, i.e., $\theta = 0$.
2. The function f decreases most rapidly when $\mathbf{u}$ is the direction of $-\nabla f$, i.e., $\theta = \pi$.
3. Any direction $\mathbf{u}$ orthogonal to a $\nabla f \neq 0$ is a direction of zero change in f.

Example 4:

1. If $f(x,y) = x^2 + y^2$, find the rate of change of f at the point $P(1,1)$ in the direction from P to $Q(2,3)$.
2. In what direction does f increase most rapidly at $(1,1)$?
3. In what direction does f decrease most rapidly at $(1,1)$?
4. What are the directions of zero change in f at $(1,1)$?

- Tangent Planes and Normal Lines
  Consider the smooth curve $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$ on the level surface $f(x,y,z) = c$. $\Rightarrow f((g(t), h(t), k(t)) = c$. Using the Chain rule to Differentiate both sides, we have

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dg}{dt}, \frac{dh}{dt}, \frac{dk}{dt} \right\rangle = 0$$

So ∇f is normal to the curve's velocity vector $d\mathbf{r}/dt$ on the level surface.

- Tangent Plane and Normal Line
  The tangent plane to the **level surface** $f(x,y,z)=c$ at the point $P_0\,(x_0,y_0,z_0)$:

$$\left.\frac{\partial f}{\partial x}\right|_{P_0}(x-x_0)+\left.\frac{\partial f}{\partial y}\right|_{P_0}(y-y_0)+\left.\frac{\partial f}{\partial z}\right|_{P_0}(z-z_0)=0$$

  The Normal line to the **level surface** $f(x,y,z)=c$ at the point $P_0\,(x_0,y_0,z_0)$:

$$x=x_0+f_x\,(P_0)\;t,\;y=y_0+f_y\,(P_0)\;t,\;z=z_0+f_z\,(P_0)\;t$$

Example 5:

Find the tangent plane and normal line of the surface $f(x,y,z)=x^2+y^2+z-4=0$ at $P_0(1,1,2)$.

- Let $F(x,y,z)=f(x,y)-z=0$. The plane tangent to a surface $z=f(x,y)$ at $P_0\,(x_0,y_0,z_0)$:

$$f_x\,(x_0,y_0)\,(x-x_0)+f_y\,(x_0,y_0)\,(y-y_0)-(z-z_0)=0.$$