14 Multiple Integrals

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Outline of Chapter 15

1. Double integrals
2. Applications of Double Integrals
3. Double Integrals in Polar Coordinates
4. Triple Integrals
5. Triple Integrals in Cylindrical and Spherical Coordinates
6. Substitution and Multiple Integrals

Definition

The **double integral** of f over the rectangle R is defined to be

$$\int\int_R f(x,y)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}, y_{ij}\right)\Delta A$$

provided that these limits exist.

- Note that the double integral is defined, based on the **double Riemann sum**.

Definition

Suppose that $f(x,y) \geq 0$. Then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \int\int_R f(x,y)dA$$

- Properties of Double Integrals:

1. Linearity:

$$\int\int_R [f(x,y)+g(x,y)]\,dA = \int\int_R f(x,y)dA + \int\int_R g(x,y)dA$$

$$\int\int_R c\,f(x,y)dA = c\int\int_R f(x,y)dA \quad \text{for constant } c$$

2. If $f(x,y) \geq g(x,y)$ for all $(x,y) \in R$, then

$$\int\int_R f(x,y)dA \geq \int\int_R g(x,y)dA$$

Example

1. Evaluate the following example:

$$\int\int_R 7dA, \quad R = [-1,1]\times[-2,5].$$

2. If $f(x,y) = k$ and $R = [a,b]\times[c,d]$, show that

$$\int\int_R k\,dA = k(b-a)(d-c)$$

Theorem

Fubini's Theorem: *If $f(x,y)$ is continuous(bounded) on the rectangle $R=[a,b]\times[c,d]$, then*

$$\int\int_R f(x,y)\,dA=\int_a^b\int_c^d f(x,y)dydx=\int_c^d\int_a^b f(x,y)dxdy.$$

- In this section we learn about how to evaluate double integrals over **bounded Nonrectangular regions**.
- There are two types of double integrals on a plane region D.

1. If $f(x, y)$ is continuous on a type I region D such that $D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$, then

$$\int\int_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx$$

2. If $f(x, y)$ is continuous on a type II region D such that $D = \{(x, y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$, then

$$\int\int_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy$$

- Note that the limits of the outer integrations must be **constants**.

- For regions that are more complicated, how do we find limits of intergation? We assume to integrate first w.r.t. y and then write x:

$$\int\int_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx$$

1. Sketch the region and write the bounded curves.
2. Find the $y-$limits of integration, using **vertical lines**. Note that $y-$limits are functions in terms of x.
3. Find the $x-$limits of integration that will be numbers.

- If you evaluate double integrals with the order of integration reversed:

$$\int\int_D f(x,y)dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dxdy,$$

then we use **horizontal lines** in step2.

Example1

1. Evaluate the iterated integral

$$\int_0^{\pi/2} \int_0^{\sin\theta} e^{\cos\theta}\, dr\, d\theta.$$

2. Evaluate the double integral $\int\int_D (2x+y)dA$, where D is the region bounded by the two parabolas $y = 2x^2$ and $y = x^2 + 4$.
3. Evaluate the following integral:

$$\int\int_D y^2 dA, \quad D = \{(x,y) \mid -1 \leq y \leq 1, -y-2 \leq x \leq y\}.$$

4. Evaluate the iterated integral

$$\int_0^1 \int_0^y y^2 e^{xy} dxdy.$$

Example2

Example2 Define the function $f(x,y)$ to be

$$f(x,y) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Then evaluate

$$\int\int_R f(x,y)dA,$$

where R is enclosed by $x-$axis and $y=x$ and the line $x=1$

- As you see the previous example, there is no general rule for choosing which order of integration will be the good one.