- For a function $f(x)$ of a single variable, we could look for local maxima and local minima, critical points, finding $x \in D$ such that $f'(x) = 0$. Note that $x \in D$ where $f'(x) = 0$ or $f'(x)$ is undefined is critical points.
- For a function $f(x, y)$ of two variables, it's a little harder to find local maxima, local minima, saddle points.

Definition

Definition of local extrema

Let $f(x, y)$ be defined on a neighborhood Ω of (a, b). Then
1. $f(a, b)$ is a local (relative) maximum value if $f(a, b) \geq f(x, y)$ for all $(x, y) \in \Omega$.
2. $f(a, b)$ is a local (relative) minimum value if $f(a, b) \leq f(x, y)$ for all $(x, y) \in \Omega$.

Theorem

First Partial Derivative test for local extrema

$f(x,y)$ has local extrema at (a,b) and f_x and f_y exist in a neighborhood Ω of (a,b) $\Rightarrow$ $f_x(a,b)=0$ and $f_y(a,b)=0$.

- The previous theorem is a sufficient but not necessary condition.

Definition

Critical Point

A point is called a critical point of f if $f_x(a,b)=0$ and $f_y(a,b)=0$ or if one or both f_x and f_y do not exist.

Example1

Find the local extrema of the function

$$f(x,y)=x^2+y^2+2x-8y+17.$$

Definition

Saddle point

A differentiable function $f(x, y)$ has a saddle point at a critical point (a, b) if in every open disk centered at (a, b) there are points (x, y) where $f(x, y) > f(a, b)$ and $f(x, y) < f(a, b)$. The point (a, b) is called a saddle point of f and the graph of f crosses its tangent plane at (a, b).

Theorem

Second partial derivatives Test for Local Extrema

Suppose that f and its first and second partial derivatives are continuous on domain D and $f_x(a, b) = f_y(a, b) = 0$.

1. f has a ***local maximum*** *at (a, b) if $f_{xx} < 0$ and $D = f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).*

2. f has a ***local minimum*** *at (a, b) if $f_{xx} > 0$ and $D = f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).*

3. f has a ***saddle point*** *at (a, b) if $D = f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).*

4. ***No conclusion*** *at (a, b) if $D = f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b).*

- In order to memorize the formula for D, it will be helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2 .$$

- Indeed, D is a determinant of the **Hessian** matrix.

Example2:

1. Find the local extrema of the function

$$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 1.$$

2. Find the local extrema or saddle points of the function $f(x,y) = x\,y$.
3. Find the local extrema and saddle points of the function

$$f(x,y) = x^4 + y^4 - 4xy + 3$$

4. Find the shortest distance from the origin to the plane $x + 2y + z = 1$.

Theorem

Extreme Value Theorem for the functions of two variables

If f is continuous on a closed, bounded set $D \subset \mathbb{R}^2$, then f attains an absolute maximum and an absolute minimum at some points.

- If f is continuous on a domain D, how can we find the absolute (global) maximum values and minimum values of a continuous function f on the **closed, bounded** domain D?

1. List critical points in D.
2. List the boundary points where f has extreme values.
3. Compare those values from 1 step and 2 step.

Example3:

1. Find the absolute maximum values and minimum values of

$$f(x,y) = 1 + 2x + 2y - x^2 - y^2$$

on the triangle in the first quadrant enclosed by the $x = 0$ and $y = 0$, $y = 4 - x$.

2. Find the absolute maximum values and minimum values of f on the set D.

$$f(x,y) = x^2 + y^2 + xy^2 + 4, \quad D = \{(x,y) \mid |x| \leq 1, |y| \leq 1\}.$$