15 Multiple Integrals

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Outline of Chapter 15

1. Double intergrals over rectangles
2. Iterated Integrals
3. Double Integrals over general regions
4. Triple Integrals
5. Change of Variables in multiple integrals.

Definition

The **double integral** of f over the rectangle R is defined to be

$$\int\int_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}, y_{ij}\right)\Delta A$$

provided that these limits exist.

- Note that the double integral is defined, based on the **double Riemann sum**.

Definition

Suppose that $f(x,y) \geq 0$. Then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \int\int_R f(x,y)dA$$

- Properties of Double Integrals:

1. Linearity:

$$\int\int_R [f(x,y)+g(x,y)]\,dA = \int\int_R f(x,y)dA + \int\int_R g(x,y)dA$$

$$\int\int_R c\,f(x,y)dA = c\int\int_R f(x,y)dA \quad \text{for constant } c$$

2. If $f(x,y) \geq g(x,y)$ for all $(x,y) \in R$, then

$$\int\int_R f(x,y)dA \geq \int\int_R g(x,y)dA$$

Example1

1. Evaluate the following example:

$$\int\int_R 7dA, \quad R = [-1,1]\times[-2,5].$$

2. If $f(x,y) = k$ and $R = [a,b]\times[c,d]$, show that

$$\int\int_R k\,dA = k(b-a)(d-c)$$

- **Iterated Integrals**: Suppose that $\int\int_R f(x,y)dxdy$ is bounded over the rectangle $R = [a,b] \times [c,d]$. Then we express the double integral to an iterated integral. You also have to understand the double integral geometrically.

1. Consider partial integration $\int_c^d f(x,y)dy$ w.r.t. y from $y = c$ to $y = d$, which depends on the value of x:

$$A(x) = \int_c^d f(x,y)dy$$

2. Integrate the function $A(x)$ w.r.t. x from $x = a$ to $x = b$:

$$\int_a^b A(x)dx = \int_a^b \int_c^d f(x,y)dydx = \int_a^b \left[\int_c^d f(x,y)dy\right] dx,$$

where the last integral is called an iterated integral.

3. Similarly,

$$\int_c^d \int_a^b f(x,y)dxdy = \int_c^d \left[\int_a^b f(x,y)dx\right] dy.$$

Example2

1. Evaluate the following double integrals

$$\int\int_R (4-2y)dA, \quad R=[0,1]\times[0,1].$$

2. Let $f(x,y)=2x-3x^2y^2$. Then find

$$\int_0^2 f(x,y)dx \text{ and } \int_0^1 f(x,y)dy.$$

Theorem

Fubini's Theorem: *If $f(x,y)$ is continuous(bounded) on the rectangle $R=[a,b]\times[c,d]$, then*

$$\int\int_R f(x,y)\,dA=\int_a^b\int_c^d f(x,y)dydx=\int_c^d\int_a^b f(x,y)dxdy.$$

Example3

Calculate the following iterated integrals

1. $\int_1^4 \int_0^1 \left(6x^2y + 2x\right) dydx$
2. $\int_1^4 \int_1^e \left(\frac{x}{y} + \frac{y}{x}\right) dydx$
3. $\int_0^1 \int_1^2 \left(x + e^{-y}\right) dxdy$