# 15 Vector Calculus

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# Outline of Chapter 15

## Example1

Examples of **a vector field.**

1. Wind velocity vectors
2. Ocean currents
3. Airflow past inclined airfoil.

## Definitions

1. $D \in \mathbb{R}^2$(plane region). A vector field on $\mathbb{R}^2$ is a function F that assigns to each point $(x, y)$ in $D$ a two-dimensional vector $\mathrm{F}(x, y)$.
2. $D \in \mathbb{R}^3$(space region) . A vector field on $\mathbb{R}^3$ is a function F that assigns to each point $(x, y, z)$ in $D$ a three-dimensional vector $\mathrm{F}(x, y, z)$.

## Example2

Find the gradient vector field of $f$.

1. $f(x, y) = ye^{xy}$ 2. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

- We start a space curve $C$ given by the parametric equations

$$x = x(t) \quad y = y(t) \quad z = z(t) \quad a \le t \le b.$$

**Definition**

If $f$ is defined on a smooth curve $C$ given by the parametric equations, then **the line integral of $f$ along $C$** is

$$\int_C f(x,y,z)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*, y_i^*, z_i^*\right) \Delta s_i$$

provided this limit exists.

- Recall the arc length

$$S(t) = \int_a^t |\mathrm{v}(\tau)|\, d\tau,$$

where $\mathrm{r}(t) = \langle g_1(t), g_2(t), g_3(t)\rangle$, $\mathrm{v}(t) = \langle g_1'(t), g_2'(t), g_3'(t)\rangle$.

- More useful formula for the line integrals is

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt.$$

- Suppose that $C$ is a piecewise-smooth curve: that is, $C = C_1 \cup C_2 \cup \cdots \cup C_n$. Then the integral of $f$ along $C$ is

$$\int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)ds + \cdots + \int_{C_n} f(x,y)ds.$$

- A vector representation of the line segment that starts at $\mathrm{r}_0$ and ends at $\mathrm{r}_1$ is given by

$$\mathrm{r}(t) = (1-t)\mathrm{r}_0 + t\mathrm{r}_1 \quad 0 \le t \le 1.$$

**Example1**

Evaluate the line integral, where $C$ is the given curve.
1. $\int_C 2yds$, $C: x = t^2, y = t$, $0 \le t \le 1$.
2 .$\int_C x^2yds$, $C$ is the upper half of the unit circle $x^2 + y^2 = 1$.

- When line integrals w.r.t. $x$ and $y$ occur together, we write

$$\int_C P(x,y)dx + \int_C Q(x,y)dy = \int_C P(x,y)dx + Q(x,y)dy.$$

**Example2**

Evaluate the line integral, where $C$ is the given curve.
1. $\int_C \left(xy^2 - \sqrt{x}\right) dy$, $C$ is the arc of the curve $y = \sqrt{x}$ from $(1,1)$ to $(9,3)$.
2. $\int_C (x+yz)dx + xdy + 2xyzdz$, $C$ consists of line segments from $(0,0,0)$ to $(1,2,-1)$ and from $(1,2,-1)$ to $(3,2,1)$.

## Definition

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve $C$ given by a vector function $\mathbf{r}(t)$ with $a \le t \le b$. Then **the line integral of $\mathbf{F}$ along $C$** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_C \mathbf{F} \cdot \mathbf{T}ds,$$

where $\mathbf{T}$ is the unit tangent vector on $C$.

## Example3

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $C$ is given by the vector function $\mathbf{r}(t)$.

1. $\mathbf{F}(x,y) = \langle xy, 3x^2 \rangle$, $\mathbf{r}(t) = \langle t^3, t \rangle$, $0 \le t \le 1$.
2. $\mathbf{F}(x,y,z) = \sin x\,\mathbf{i} + \cos y\,\mathbf{j} + xy\,\mathbf{k}$, $\mathbf{r}(t) = t\,\mathbf{i} - t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $0 \le t \le 1$.