15.2 Double integrals over general regions

- In this section we learn about how to evaluate double integrals over **bounded Nonrectangular regions**.
- There are two types of double integrals on a plane region D.

1. If $f(x,y)$ is continuous on a type I region D such that $D = \{(x,y) \mid a \leq x \leq b,\, g_1(x) \leq y \leq g_2(x)\}$, then

$$\int\int_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dydx$$

2. If $f(x,y)$ is continuous on a type II region D such that $D = \{(x,y) \mid c \leq y \leq d,\, h_1(y) \leq x \leq h_2(y)\}$, then

$$\int\int_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dxdy$$

- Note that the limits of the outer integrations must be **constants**.

- For regions that are more complicated, how do we find limits of intergation? We assume to integrate first w.r.t. y and then write x:

$$\int\int_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx$$

1. Sketch the region and write the bounded curves.
2. Find the $y-$limits of integration, using **vertical lines**. Note that $y-$limits are functions in terms of x.
3. Find the $x-$limits of integration that will be numbers.

- If you evaluate double integrals with the order of integration reversed:

$$\int\int_D f(x,y)dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\,dxdy,$$

then we use **horizontal lines** in step2.

Example1

1. Evaluate the iterated integral

$$\int_0^{\pi/2} \int_0^{\sin\theta} e^{\cos\theta} \, dr \, d\theta.$$

2. Evaluate the double integral $\int\int_D (2x+y)dA$, where D is the region bounded by the two parabolas $y = 2x^2$ and $y = x^2 + 4$.
3. Evaluate the following integral:

$$\int\int_D y^2 dA, \quad D = \{(x,y) \mid -1 \le y \le 1, -y-2 \le x \le y\}.$$

4. Evaluate the iterated integral

$$\int_0^1 \int_0^y y^2 e^{xy} \, dxdy.$$

Example2

Example2 Define the function $f(x, y)$ to be

$$f(x,y)=\begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$

Then evaluate

$$\int\int_R f(x,y)dA,$$

where R is enclosed by $x-$axis and $y = x$ and the line $x = 1$

- As you see the previous example, there is no general rule for choosing which order of integration will be the good one.