Definition

The triple integral of f over the box B is

$$\iiint_B f(x,y,z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right) \Delta V,$$

if this limit exists.

- The right side of the equation in the previous definition is called **the triple Riemann sum**. The sample point $\left(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*\right)$ is in sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

and each sub-boxes has the volume $\Delta V = \Delta x \Delta y \Delta z$.

Theorem

Fubini's Theorem for Triple Integrals

If f is a continuous(bounded) on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\begin{aligned}\iiint_B f(x,y,z)dV &= \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz \\ &= \int_c^d \int_r^s \int_a^b f(x,y,z)dxdzdy \\ &= \cdots\end{aligned}$$

Example1

Evaluate the triple integral, integrating w.r.t $y \Rightarrow z \Rightarrow x$

$$\iiint_B xyz^2 dV,$$

where B is the rectangle box given by $B = \{(x, y, z) \mid 0 \leq x \leq 1, -1 \leq y \leq 2, 0 \leq z \leq 3\}$.

Example2

Evaluate the iterated(triple) integral.

(1)

$$\int_0^1 \int_0^z \int_0^{y-z} x - 2y \, dx \, dy \, dz$$

(2)

$$\int_0^{\pi/2} \int_0^1 \int_0^{\sqrt{1-z^2}} z \cos x \, dy \, dz \, dx$$

- There are two types of triple integrals on a solid region E.

1. If f is continuous(bounded) on **type I** solid region $E = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$, then

$$\int\int\int_E f(x,y,z)dV = \int\int_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\right] dA$$
$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdydx,$$

   where the projection D of E onto the $xy-$plane is $D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$.

2. Similarly, if f is continuous(bounded) on **type II** solid region $E = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$, then

$$\int\int\int_E f(x,y,z)dV = \int\int_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\right] dA$$
$$= \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dzdxdy,$$

   where $D = \{(x,y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$.

- To evaluate the triple integral $\int\int\int_E f(x,y,z)dV$ over E,

1. Integrate it w.r.t. $z \Longrightarrow y \Longrightarrow x$ or $(z \Longrightarrow x \Longrightarrow y)$
2. Sketch a solid region and label limits. It is not easy!

Example3

Evaluate the triple integral.

1.

$$\int\int\int_E y\,dV,$$

where $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, x-y \le z \le x+y\}$.

2.

$$\int\int\int_E x\,y\,dV,$$

where E lies under the plane $z = 1+x+y$ and above the region in the $xy-$plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$.