- We recall a change of variable (a substitution rule) for the single integrals. If $x = g(u)$ is differentiable and f is integrable, then

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)du,$$

  where $a = g(c)$ and $b = g(d)$. Here we used $x-$substitution rather than $u-$substitution.

- For the double integrals we consider a change of variable that is given by a transformation $T : S \to R$ from the $uv-$plane to the $xy-$plane:

$$T(u,v) = (x,y),$$

  where $x = g(u,v)$ and $y = h(u,v)$. We usually assume that $g, h \in C^1$.

Example1

Find the image of the set S under the given transformation.

1. $S = \{(u, v) \mid 0 \le u \le 1, 0 \le v \le 1\}$; $x = u^2 - v^2$, $y = uv$
2. $S = \{(u, v) \mid 0 \le u \le 2, 0 \le v \le 3\}$; $x = 2u + 3v$, $y = u - v$
3. S is the triangular region with vertices $(0, 0)$, $(1, 1)$, $(0, 1)$; $x = u$, $y = v^2$

Theorem

Change of Variables in a Double Integral: *Suppose that T is $1-1$ transformation. Letting the substitution $x = g(u, v)$ and $y = h(u, v)$, we can set up*

$$\int\int_R f(x, y)dxdy = \int\int_S f(g(u, v), h(u, v))\,|J(u, v)|\,dudv,$$

where the Jacobian $J(u, v)(= \partial(x, y)/\partial(u, v))$ is

$$J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

- How to use the given transformation to evaluate the double integrals.

1. Solve the transformation for x and y. This step may omit if the transformation is given by x and y.
2. Sketch the region R (image of S) on the $xy-$plane and find the boundary of R.
3. Using the substitution, find the boundary of S and sketch the region S(pre-image of R) on the $uv-$plane.
4. Find the Jacobian (determinant) from step 1.
5. Use the formula(derived by the substitution) in the Theorem.

Example2

1. Find the Jacobian of the transformation.

(1) $x = 4u - v,\ y = u + 2v$

(2) $x = r\cos\theta,\ y = r\sin\theta$

(3) $x = e^{-r}\sin\theta,\ y = e^{r}\cos\theta$

2. Use the given transformation to evaluate the double integral.

(1) $\int\int_R (x - 3y)dA$, where R is the triangular region with vertices $(0,0), (2,1)$, and $(1,2)$; $x = 2u + v,\ y = u + 2v$.

(2) $\int\int_R xydA$, where R is the region in the first quadrant bounded by the lines $y = x$ and $y = 4x$ and the hyperbolas $xy = 1,\ xy = 4$; $x = u/v,\ y = v$.

3. Evaluate

$$\int\int_R \sqrt{x+y}(y-2x)^2 dA$$

where R is the triangular region with $x-$axis and $y-$axis and the line $y = -x + 1$. Note that you need to take suitable substitutions.