16 Vector Calculus

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Outline of Chapter 16

Example1

Examples of **a vector field.** 1. Wind velocity vectors
2. Ocean currents 3. Airflow past inclined airfoil.

Definitions

1. $D \in \mathbb{R}^2$(plane region). A vector field on $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.
2. $D \in \mathbb{R}^3$(space region) . A vector field on $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in D a three-dimensional vector $\mathbf{F}(x, y, z)$.

Example2

1. Sketch the vector field $\boldsymbol{F}$ by drawing a diagram.
(1) $\boldsymbol{F}(x, y) = -\boldsymbol{i} + \boldsymbol{j}$ (2) $\boldsymbol{F}(x, y) = \boldsymbol{i} + (x - y)\boldsymbol{j}$
2. Find the gradient vector field of f.
(1) $f(x, y) = ye^{xy}$ (2) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

- We start a space curve C given by the parametric equations

$$x = x(t) \quad y = y(t) \quad z = z(t) \quad a \leq t \leq b.$$

Definition

If f is defined on a smooth curve C given by the parametric equations, then **the line integral of f along C** is

$$\int_C f(x, y, z) ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*) \Delta s_i$$

provided this limit exists.

- Recall the arc length

$$S(t) = \int_a^t |\mathbf{v}(\tau)| \, d\tau,$$

where $\mathbf{r}(t) = \langle g_1(t), g_2(t), g_3(t) \rangle$, $\mathbf{v}(t) = \langle g_1'(t), g_2'(t), g_3'(t) \rangle$.

- More useful formula for the line integrals is

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt.$$

- Suppose that C is a piecewise-smooth curve: that is, $C = C_1 \cup C_2 \cup \cdots \cup C_n$. Then the integral of f along C is

$$\int_C f(x,y)ds = \int_{C_1} f(x,y)ds + \int_{C_2} f(x,y)ds + \cdots + \int_{C_n} f(x,y)ds.$$

- A vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \le t \le 1.$$

Example1

Evaluate the line integral, where C is the given curve
1. $\int_C 2yds$, $C: x = t^2, y = t$, $0 \le t \le 1$.
2 .$\int_C x^2yds$, C is the upper half of the unit circle $x^2 + y^2 = 1$.

- When line integrals w.r.t. x and y occur together, we write

$$\int_C P(x,y)dx + \int_C Q(x,y)dy = \int_C P(x,y)dx + Q(x,y)dy.$$

Example2

Evaluate the line integral, where C is the given curve.
1. $\int_C \left(xy^2 - \sqrt{x}\right) dy$, C is the arc of the curve $y = \sqrt{x}$ from $(1,1)$ to $(9,3)$.
2. $\int_C (x+yz)dx + xdy + 2xyzdz$, C consists of line segments from $(0,0,0)$ to $(1,2,-1)$ and from $(1,2,-1)$ to $(3,2,1)$.

Definition

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$ with $a \le t \le b$. Then **the line integral of F along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_C \mathbf{F} \cdot \mathbf{T} ds,$$

where $\mathbf{T}$ is the unit tangent vector on C.

Example3

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.
1. $\mathbf{F}(x,y) = \langle xy, 3x^2 \rangle$, $\mathbf{r}(t) = \langle t^3, t \rangle$, $0 \le t \le 1$.
2. $\mathbf{F}(x,y,z) = \sin x\,\mathbf{i} + \cos y\,\mathbf{j} + xy\,\mathbf{k}$, $\mathbf{r}(t) = t\,\mathbf{i} - t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $0 \le t \le 1$.