- Recall the F. T. of Calculus (II).

Theorem

Let C be a smooth curve given by the ***vector function*** $\mathbf{r}(t)$ *with* $a \leq t \leq b$. *If f is smooth on C, then*

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

- **Independence of Path**

1. Suppose that any two paths C_1 and C_2 in the domain D have the same initial and terminal point. Then one implication of the previous Theorem is

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

2. If $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ then the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is called **independent of path**. In general, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

- A vector field $\mathbf{F}$ is called **a conservative vector field** if $\exists f$ such that $\mathbf{F} = \nabla f$. f is called a **potential function** of $\mathbf{F}$.

Theorem

*Suppose that C is any **closed** path (its terminal point coincides with its initial point) in D.*
Then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in $D \Leftrightarrow \int_C \mathbf{F} \cdot d\mathbf{r} = 0$.

- In the next theorem, we see that the only vector functions(fields) that are independent of path are conservative.

Theorem

1. Suppose that $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D, i.e., $\exists f$ such that $\nabla f = \mathbf{F}$.
2. If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field and P, Q are smooth on a domain, then we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \textit{throughout } D.$$

Theorem

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} = \langle P, Q \rangle$ be a vector on an open simply-connected region D. Suppose that P, Q are smooth and $\partial P/\partial y = \partial Q/\partial x$ throughout D. Then $\mathbf{F}$ is conservative.

Example1

Determine whether or not the following vector fields are conservative.

1. $\mathbf{F}(x, y) = (x - 2y)\mathbf{i} + (2x + 1)\mathbf{j}$ 2. $\mathbf{F}(x, y) = \left\langle 1 + 4xy, 2x^2 - 5y \right\rangle$

Example2

Determine whether or not the vector field $\mathbf{F} = \left\langle 1 + 2xy, x^2 - 3y^2 \right\rangle$ is conservative.

1. If it is, find a function f such that $\mathbf{F} = \nabla f$.
2. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by

$$\mathbf{r}(t) = \left\langle e^t \sin t, e^t \cos t \right\rangle, \quad 0 \leq t \leq \pi.$$

Example3

1. Find a function f such that $\mathbf{F} = \nabla f$.
2. Use part 1 to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

$$\mathbf{F}(x, y, z) = \langle yz, xz, xy + 2z \rangle,$$

where C is the line segment from $(1, 0, -2)$ to $(4, 6, -3)$.

- Example3 will be done in Section16.5.

Example4

1. Find a function f such that $\mathbf{F} = \nabla f$.
2. Use part 1 to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

$$\mathbf{F}(x, y) = \left\langle xy^2, x^2y \right\rangle,$$

where C : $\quad \mathbf{r}(t) = \left\langle t + \sin\frac{\pi t}{2}, t + \cos\frac{\pi t}{2} \right\rangle, \quad 0 \le t \le 1.$