- Green's Theorem provides the relationship between a $\int_C$ around a simple closed curve C and a $\int\int_D$ over the plane region D bounded by C.
- For the convention the **positive orientation of** a simple closed curve C will be a single counterclockwise traversal of C.
- If C is the given $\mathbf{r}(t)$, $a \leq t \leq b$, then D is always on the left as the point $\mathbf{r}(t)$ traverses C.

Theorem

Green's Theorem:

Let C be a positively oriented, piecewise-smooth, simple-closed curve in the plane and D be the region bounded by C. If P and Q are smooth on an open region that contains D, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$

where $\mathbf{F} = \langle P, Q \rangle$.

Example1

Evaluate the line integral by two methods: (1) directly and (2) using Green's Theorem

$$\int_C 4x^3 dx + 2xydy,$$

where C is the triangular curve consisting of the line segments (0,0) to (1,0), from (1,0) to (0,1), and from (0,1) to (0,0).

Example2

Evaluate the line integral by two methods: (1) directly and (2) using Green's Theorem

$$\oint_C (x-y)dx + (x+y)dy,$$

C is the circle $x^2 + y^2 = 4$.

- Let D be the region bounded by a simple-closed curve C. Then Green's Theorem provides the following formula for the area of D:

$$A = \oint_C xdy = -\oint_C ydx = \frac{1}{2}\oint_C xdy - ydx$$

Example3

Find the area enclosed by (1) the ellipse $x^2/a^2 + y^2/b^2 = 1$ and (2) the circle $x^2 + y^2 = r^2$.

Example4

Use Green's theorem to evaluate the line integral along the given positively oriented curve.

$$\int_C y^3 dx - x^3 dy,$$

where C is the circle $x^2 + y^2 = 4$.

Example5

Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$$\mathbf{F}(x, y) = \left(e^x + x^2 y\right)\mathbf{i} + \left(e^y - xy^2\right)\mathbf{j},$$

where C is the circle $x^2 + y^2 = 4$ oriented clockwise.