16.5 Curl and Divergence

- In this section we define two operators; one (curl) produces a vector field and the other (divergence) produces a scalar field.
- Those operators are defined, based on the del operator:

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle.$$

- Suppose that $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field on $\mathbb{R}^3$. Then

1. curl$\mathbf{F} = \nabla \times \mathbf{F}$
2. div$\mathbf{F} = \nabla \cdot \mathbf{F}$

Theorem

1. If f is a smooth function with three variables, then

$$curl(\nabla f) = \mathbf{0}$$

2. If $\mathbf{F}$ is a smooth vector field on $\mathbb{R}^3$, then

$$div\ curl\mathbf{F} = 0$$

Example1

Find (1) the curl (2) the divergence of the vector field.

1. $\mathbf{F}(x,y,z) = xyz\,\mathbf{i} - x^2y\mathbf{k}$
2. $\mathbf{F}(x,y,z) = \langle \ln x, \ln(x\,y), \ln(x\,y\,z)\rangle$

Theorem

If $\mathbf{F}$ *is a smooth vector field on* $\mathbb{R}^3$ *and curl* $\mathbf{F} = \mathbf{0}$, *then* $\mathbf{F}$ *is a conservative vector field.*

Example2

Determine whether or not the vector field is conservative. If it is conservative, find f such that $\mathbf{F} = \nabla f$ and evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C:

1.$\mathbf{F}(x,y,z) = \left\langle y^2z^3, 2xyz^3, 3xy^2z^2 \right\rangle, C : \mathbf{r}(t) = \left\langle t, t^2, t^3 \right\rangle, 0 \leq t \leq 1.$

2.$\mathbf{F}(x,y,z) = ye^{-x}\,\mathbf{i} + e^{-x}\mathbf{j} + 2z\,\mathbf{k}, C : \mathbf{r}(t) = \langle t, e^t, \ln t \rangle, 1 \leq t \leq e.$

- See the textbook for vector forms of Green's Theorem