16.7 Surface Integrals

- Suppose that a surface S has a vector equation

$$\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \quad \text{for } (u, v) \in D,$$

where the parameter domain D is a rectangle. Then **the surface integral of f over the surface S** is defined to be

$$\int\int_S f(x, y, z) dS = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(P_{ij}^*\right) \Delta S_{ij}.$$

- The approximation for the patch area ΔS_{ij} is

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v,$$

where

$$\mathbf{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle, \quad \mathbf{r}_u = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$$

- **The surface integral of f over the surface S is**

$$\int\int_S f(x,y,z)dS = \int\int_D f(\mathbf{r}(x,y))\left|\mathbf{r}_x \times \mathbf{r}_y\right| dA,$$

where D is the parameter domain and $f(\mathbf{r}(x,y))$ is evaluated by writing $x = x,\ y = y,\ z = g(x,y)$.

- The more useful formula for the surface integral is

$$\int\int_S f(x,y,z)dS = \int\int_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}dA,$$

where $z = g(x,y)$ and D is the projection of the surface S on the $xy-$plane.

Example1

Evaluate the following surface integral.
1. $\int\int_S x^2 y\, z\, dS$, where S is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $[0,3] \times [0,2]$.
2. $\int\int_S x^2 z^2 dS$, where S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 3$.

Definition

If **F** is a continuous vector field defined on an oriented surface S with unit normal vector **n**, then the surface integral of **F** over S is

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS.$$

This integral is also called the **flux** of **F** across S.

- If $\mathbf{F} = \langle P, Q, R\rangle$ and $z = g(x, y)$, the surface integral of **F** over S(flux of **F**) becomes

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS = \int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA.$$

Example2

Evaluate the surface integral $\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS$ for the given vector field **F** and the oriented surface S: $\mathbf{F} = \langle xze^y, -xze^y, z\rangle$, S is the part of the plane $x + y + z = 1$ in the first octant and has downward orientation.