16.8 Stokes' Theorem

- Green's Theorem can be generalized to Stokes' Theorem.
- Whereas **Green's Theorem** relates $\int\int_D$ over a plane(flat) region D to a line integral around its plane boundary curve, **Stokes' Theorem** relates a surface integral over surface (not flat) S to a line integral around the boundary curve of S.
- Letting $\mathbf{F} = \langle P, Q, 0 \rangle$, we recall Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int\int_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA.$$

Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components are smooth functions on an open region in $\mathbb{R}^3$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, dS$$

- Once you find $\nabla \times \mathbf{F} = \langle P, Q, R \rangle$ and a graph (a surface) $z = g(x, y)$, you have to use the following:

$$\int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS = \int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA,$$

where D is the parameter domain.

Example

1. Use Stokes' Theorem to evaluate $\int\int_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, dS$.
$\mathbf{F}(x,y,z) = \langle ye^z, x\sin z, xy\cos z \rangle$, S is the hemisphere $z^2 + x^2 + y^2 = 16$, $z \geq 0$, oriented int the direction of the positive z-axis.
2. Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.
$\mathbf{F} = \left\langle x + y^2, y + z^2, z + x^2 \right\rangle$ C is the triangle with vertices $(1,0,0), (0,1,0), (0,0,1)$ and oriented counterclockwise as viewed from above.