16.9 The Divergence Theorem

Theorem

Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a smooth vector field on an open region that contains E. Then

$$\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS = \int\int\int_E \nabla \cdot \mathbf{F}\, dV.$$

Example

1. Verify that the Divergence Theorem is true for $\mathbf{F}$ on the region E: $\mathbf{F}(x, y, z) = \langle 3x, xy, 2xz \rangle$ and E is the cube bounded by planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$.
2. Use the Divergence Theorem to calculate the surface integral $\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS$; that is, calculate the flux of $\mathbf{F}$ across S: $\mathbf{F}(x, y, z) = \langle \cos z + xy^2, xe^{-z}, \sin y + x^2 z \rangle$ and S is the surface of the solid bounded by the parabolid $z = x^2 + y^2$ and the plane $z = 4$.