1. Introduction

Department of Mathematics & Statistics

ASU

Outline of Chapter 1

1. Chapter 1: Introduction to DEs
   - Applications Leading to DEs
   - Basic Concepts
   - Direction Fields for the First Order Equations

1.1 Applications Leading to DEs

- Mathematical models are constructed, when physical or real life application problems are understood, where mostly changing quantities are involved.
- The mathematical models provide equations containing derivatives (rates of change) which are called DEs. For example, find a function $y = y(t)$ satisfying

$$\frac{dy}{dt} = y.$$

- **Solving DEs** means finding a **function** satisfying conditions.

Mathematical Models

- Population Growth and Decay
- Newton's Laws
- Glucose Absorption by the Body
- Spread of Epidemics or COVID-19

1.2 Basic Concepts

- **The 17th century:** the subject of DEs is originally studied by Newton and Leibniz. Newton classified first order differential equations;

$$y' = f(x, y).$$

  Leibniz initially used the notation dy/dx and developed methods to solve DEs. This first order DE is a simplest one but so many mathematical ideas are in it. The given function $f(x, y)$ is called the rate function.

- Initial Value Problems (IVPs) consist of DEs and initial data:

$$\begin{cases} y' = f(x, y), \\ y(a) = b. \end{cases}$$

- We will investigate solvablilites of the IVPs.

There are several useful ways to classify DEs.

1. ODEs and PDEs

- **ODEs:** unknown functions (solutions) depend on a **single** variable and only ordinary derivatives appear. For example,

$$u'(t) = \alpha\, u^2(t), \quad \text{where } \alpha \text{ is a constant.}$$

- **PDEs:** unknown functions depend on more than two variables and partial derivatives are in the PDEs. For example,

$$u_{tt}(t,x) = c\, u_{xx}(t,x), \quad \text{where } c \text{ is a constant.}$$

2. Systems of DEs: for example,

$$\frac{dx}{dt} = x + xy,$$
$$\frac{dy}{dt} = y - xy,$$

where $x = x(t)$ and $y = y(t)$ are solutions dependent of t.

3. Order

- **Its definition:** the order of DEs is the order of the highest derivative that appears in the equation. For example, we can consider the third order ODEs;

$$u'''(t) - u'(t) + u(t) = 0.$$

- More general ODEs with nth order can be written to be

$$F\left(t, u(t), u'(t), \cdots, u^{(n)}(t)\right) = 0. \qquad (1)$$

  Note that y is mostly used in the textbook, instead of u. We may consider the following example of a fourth order DE;

$$y^{(4)} + 2ty' + yy' = t^2. \qquad (2)$$

- In order to avoid the ambiguity of the general form of ODE (1), we assume that the higher derivative can be expressed explicitly in terms of lower derivatives;

$$u^{(n)} = f\left(t, u, u', \cdots, u^{(n-1)}\right)$$

  which will be studied in our DE class.

4. Linear and Nonlinear

- The general linear ODE of order n is

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)\,y = g(t), \qquad (3)$$

  where $a_i(t)$ for all $1 \leq i \leq n$ and $g(t)$ are given.
- An equation that is not of the form (3) is called a nonlinear ODE. The ODE (2) is a good example of nonlinear ODE.
- Nonlinear ODEs are, in general, more difficult to solve.

From the mathematical point of view

1. We need to prove the existence of solutions and uniqueness.
2. Once this mathematical task is complete (at least showing the existence of solutions), then we can compute reasonable approximations by using numerical methods.
3. In general, we cannot solve ODEs or (PDEs) by hands. Numerical methods are highly recommendable.

1.3 Direction Fields for the First Order Equations

- **Direction fields (slope fields)** are helpful for us to guess the solutions of first order DEs in the form

$$\frac{dy}{dt} = f(t, y).$$

1. When we sketch them, a short line segment whose slope is the value of f at each point of the grid is drawn at that point. Thus, each line segment is tangent to the graph of the solution. When DEs are quite difficult to solve, constructing a direction field may be useful to understand them.
2. Mathematical softwares such as *Mathematica* or *Maple* are usually used to draw a direction field for complicated rate functions.

- Finding an **equilibrium solution** is very important to sketch direction fields. Taking $f(t,y)=0$, we can find it. So the equilibrium solution will be a horizontal line.

Example

Draw a direction field for the following given differential equation. Determine the behavior of the solution y as $t \to \infty$.

1. $y' = -1 - 2y$.
2. $y' = y + 1$.
3. $-y' = y - 2$.
4. $2y + 2 = 4y'$.
5. $y' + y = 1$.