2.3 Existence and uniqueness of solutions of linear and nonlinear equations

- **Existence and uniqueness:** Does every **IVP** have exactly one solution? The question will be answered by the following Theorems.

1. For the first order **linear** DEs, we consider the following fundamental Theorem (1).

Theorem

If the functions $p, g \in C(I)$ *with* $x_0 \in I$ *for an open interval* I*, then* $\exists!$ *solution* $y = y(x)$ *that satisfies the DE*

$$y' + p(x)\, y = g(x)$$

for each $x \in I$ *and that also satisfies the initial condition* $y(x_0) = y_0$*, where* y_0 *is an initial data.*

2. For the first order **nonlinear** DEs, the following Theorem (see the textbook pp. 56) will be considered.

Theorem

Assume that $f, \partial f/\partial y \in C\left((a, b) \times (c, d)\right)$ *with* $(x_0, y_0) \in (a, b) \times (c, d)$*. Let* $h > 0$*. Then in some interval* $(x_0 - h, x_0 + h) \subset (a, b)$, $\exists!$ *solution* $y = y(x)$ *of IVP*

$$\frac{dy}{dx} = f(x, y(x)), \quad y\left(x_0\right) = y_0.$$

- The proof for the Theorems requires great depth. So you may refer to advanced DEs books.
- Here is an **important remark:** the existence of solutions can be guaranteed on the basis of **the continuity of** f **alone.**