2.5 Exact Equations

- In this section, we consider a class of equations known as **exact equations**. Those equations which can be solved by elementary integration methods are rather special; most first order DEs cannot be solved in this way.
- **Exact DEs** have the following form;

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0. \tag{1}$$

Suppose that there is a function $F(x,y)$ such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y), \quad \frac{\partial F(x,y)}{\partial y} = N(x,y).$$

Then it follows from (1) that

$$0 = M(x,y) + N(x,y)\frac{dy}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = \frac{d}{dx}F(x,y(x)).$$

Thus solutions y are given implicitly by

$$F(x,y) = c, \quad \text{where c is an arbitrary constant.}$$

- In order to consider systematic way of determining whether a given DE is exact, we provide the following Theorem.

Theorem

*Let the functions $M, N, M_y,$ and, $N_x \in C((\alpha, \beta) \times (\gamma, \delta))$. Then, the equation $M(x, y) + N(x, y)y' = 0$ is an **exact** DE in an open rectangular region $(\alpha, \beta) \times (\gamma, \delta) \iff M_y(x, y) = N_x(x, y)$ at each point of the rectangular region. That is, $\exists$ a function F such that $F_x(x, y) = M(x, y), \quad F_y(x, y) = N(x, y) \iff M$ and N satisfy the equation $M_y(x, y) = N_x(x, y)$.*

- From the previous theorem, solutions of exact DEs are given **implicitly** by

$$F(x, y(x)) = c,$$

where c is arbitrary.