3.1 Numerical Methods: the Explicit and Implicit Euler Method

- Recall the first order IVP

$$\frac{dy}{dt} = f(t, y(t)), \quad y(t_0) = y_0.$$

Then there are two important facts;

1. If $f, \partial f/\partial y \in C$, then the IVP has a unique solution y in some interval I containing t_0.
2. However, it is **always not possible** to find the solution y by symbolic manipulation of the DE.

- When f is **not a simple** function, numerical methods are highly recommendable to obtain approximations; the popular methods are **the Explicit Euler's method (forward) and the Implicit Euler method (backward).**

The Euler Methods

- We partition the interval $[a, b]$ uniformly:

$$a = t_0 < t_1 < \cdots < t_n \cdots < t_{m-1} < t_m = b,$$

where $h = t_{n+1} - t_n$ with $n \geq 0$ is the size of the subintervals and $m = (b-a)/h$ is the number of the subintervals. Let w_n be an **approximation** at t_n for $n \geq 0$. Assume that $w_0 = y_0$.

1. **Explicit (Forward)**
   We can set up the iterative formula:

$$w_{n+1} = w_n + h f\left(t_n, w_n\right).$$

2. **Implicit (Backward)**
   Its iterative formula is

$$w_{n+1} = w_n + h f\left(t_{n+1}, w_{n+1}\right).$$

- Note that the Implicit Method is stable numerically but solving the next step solution will be hard, depending on what type of the function f we have.

Example

- Consider the following IVP: for $t \in [0,1]$

$$\begin{cases} \frac{dy}{dt} = 2 - t + y, \\ y(0) = 1. \end{cases}$$

  Then find the exact solution. Use the explicit and implicit Euler's iterative formula to find the first three approximations with $h = 0.01$. Estimate errors.

The following table shows the approximations and errors. E_I is an error by the implicit Euler method and E_E is an error by the explicit Euler method.

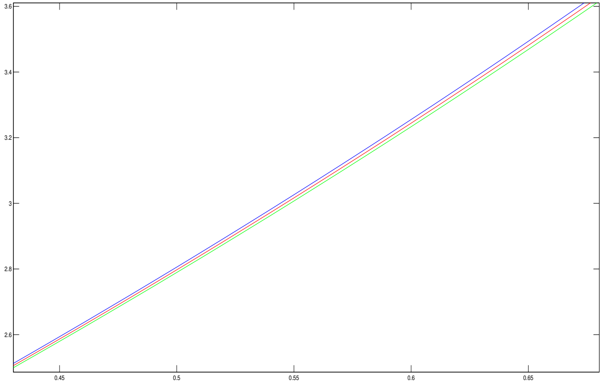
Table: Errors at the first three points

	$t = 0.01$	$t = 0.02$	$t = 0.03$	$t = 1$
E_I	0.0001	0.0002	0.0003	0.0026
E_E	0.0001	0.0002	0.0003	0.0030

Answer for the example

- Note that the exact solution is $y = 2e^t + t - 1$. (Red: the exact solution, Green: the approximation by the Explicit, Blue: the approximation by the Implicit).

Figure: Comparison



HW assignments

- When you compute approximations at the first three steps, use your graphing calculator.
- Consider the following IVPs: for $t \geq 0$

$$(1)\begin{cases} \frac{dy}{dt} = y, \\ y(0) = 1. \end{cases} \qquad (2)\begin{cases} \frac{dy}{dt} - y = 1, \\ y(0) = 1. \end{cases}$$

$$(3)\begin{cases} 2\frac{dy}{dt} = y - \frac{1}{2}, \\ y(0) = 1. \end{cases} \qquad (4)\begin{cases} (t+1)\frac{dy}{dt} = y, \\ y(0) = 1. \end{cases}$$

$$(5)\begin{cases} \frac{dy}{dt} = (2t+1)y, \\ y(0) = 1. \end{cases} \qquad (6)\begin{cases} \frac{dy}{dt} - y = \cos t, \\ y(0) = 1. \end{cases}$$

Then find the exact solution. Use the explicit and implicit Euler's iterative formula to find the first three approximations with $h = 0.01$. Estimate errors.