5. Linear Second Order DEs

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Outline of Chapter 5

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2. Constant Coefficient Homogeneous Equations
3. Nonhomogenous Linear Equations
4. The Method of Undetermined Coefficients I
5. The Method of Undetermined Coefficients II
6. Reduction of Order
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- There are a couple of **main reasons** to study second order linear DEs.

1. They have abundant theoretical structures that form the foundation of many systematic methods of solution.
2. They play critical and fundamental roles in investigating mathematical physics, e.g., fluid mechanics, heat conduction, wave motion, ...

Homogeneous Linear Equations

- A second order ODE has the following form

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right), \tag{1}$$

where f is a given function. Usually, the **independent** variable is denoted by t, x and y is used to designate the **dependent** variable. The DE (1) is called a second order linear ODE if

$$f\left(x, y, y'\right) := g(x) - p(x)\, y' - q(x)y, \tag{2}$$

where g, p, and q are specified function of x but are not dependent of y.

- From (1) and (2), the **standard form** of a second order linear ODE can be rewritten as

$$y'' + p(x)y' + q(x)y = g(x). \tag{3}$$

A second order linear ODE (3) is said to be homogeneous if $g(x) = 0$. Otherwise, (3) is called nonhomogeneous.

- The following theorem **5.1.6** (see pp.202) is the summary of this section.

Theorem

Suppose $p, q \in C([a,b])$ *and let* y_1 *and* y_2 *be solutions of*

$$y'' + p(x)y' + q(x)y = 0 \quad \text{on } (a, b). \tag{4}$$

Then the following statements are equivalent.
(a) The general solution of (4) is $y = c_1 y_1 + c_2 y_2$*.*
(b) $\{y_1, y_2\}$ *is a fundamental set of solutions of (4).*
(c) y_1, y_2 *are linearly independent.*
(d) The Wronskian $W(x)$ *of* $\{y_1, y_2\}$ *is nonzero at all points in* (a, b)*, where*

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$