5.3–5.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

- Return to the nonhomogeneous second order linear DE

$$y'' + p(x)y' + q(x)y = g(x), \qquad (1)$$

where $p, q, g \in C(I)$ with the open interval I.

Theorem

The general solution of the nonhomogeneous equation (1) can be written in the following form

$$y = c_1 y_1(x) + c_2 y_2(x) + Y(x),$$

where $\{y_1, y_2\}$ is a fundamental set of solutions of the ***homogeneous*** *equation and Y is some specific solution of nonhomogeneous equation (1) which is called a particular solution of (1).*

How to solve a nonhomogeneous Eq.

- By the Theorem, we go through three steps to solve.

1. Find the general solution y_c (called complementary solution) of the corresponding homogeneous Eq.
2. Find a particular solution $Y(x)$ of the nonhomogeneous Eq.
   (1) **How to find it**?: we make an initial guess about the form of the particular solution $Y(t)$. (Method of Undet. Coeffi.)
   (2) How to set up the initial guess?

	$g(x)$	$Y(x)$
	e^{ax}	Ae^{ax}
	$\sin bx$ or $\cos bx$	$A\sin bx + B\cos bx$
	$e^{ax}\sin bx$ or $e^{ax}\cos bx$	$e^{ax}(A\sin bx + B\cos bx)$
	polynomials with the power n	$a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$

   Then we substitute $Y(x)$ into the nonhomogeneous Eq. and determine the constants.
3. The solution of Eq (1) is $y(x) = y_c(x) + Y(x)$, where y_c comes from the step 1 and Y from step 2.

Examples1

- Suppose that $y'' + p(x)y' + q(x)y = g_1(x) + g_2(x)$ and $Y_1(x)$ and $Y_2(x)$ are particular solutions of the equation $y'' + p(x)y' + q(x)y = g_1(x)$ and $y'' + p(x)y' + q(x)y = g_2(x)$, respectively. Then, $Y_1(x) + Y_2(x)$ is a particular solution of $y'' + p(x)y' + q(x)y = g_1(x) + g_2(x)$.

1. Find a **particular** solution of
   (1) $y'' - 3y' + 2y = 2e^{3x}$ (2) $y'' + y' - 6y = x$.
2. Find the **general** solution of
   (1) $y'' - 3y' - 4y = 2\cos x$ (2) $y'' - 2y' + 5y = 3e^{2x}$.
3. Find the solution of the IVP:
   $y'' - 2y' - 3y = 3x, \quad y(0) = 5/3, \quad y'(0) = 6.$
4. Find a particular solution of $y'' - 3y' - 4y = 3e^{2x} + 2\sin x$.
5. Find a particular solution of $y'' - 3y' - 4y = -6e^{x} + x$.

Examples2

1. Find **the general** solution of
   (1) $y'' - 3y' + 2y = e^{-x}$ (2) $y'' + 2y' + y = \sin x$
2. Find the solution of the IVP:
   $y'' - 4y' - 12y = 12x + 16, \quad y(0) = 1, \quad y'(0) = 2.$
3. Find the solution of the IVP:
   $y'' - 4y' + 4y = \cos 2x, \quad y(0) = 1, \quad y'(0) = 0.$
4. Find the solution of the IVP:
   $y'' + y = e^{x/2}, \quad y(0) = 1, \quad y'(0) = -1.$
5. Find a particular solution of $y'' - 3y' - 4y = 2e^{-x}$.

Note that the the method of undetermined coefficients do not work for the problem (#5).