# 10 Linear systems of DEs

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ASU

1. **Introduction to systems of DEs**
2. **Linear systems of DEs**

- Systems of DEs involving more than one unknown functions (solutions) arise in many practical physical applications.

# 10.2 Linear systems of DEs

- We assume that there are $n$ number of unknown functions (solutions), i.e., $y_1(t), y_2(t), \cdots, y_n(t)$. Then a first order linear system of DEs can be written in the form

$$\begin{aligned} y_1 &= a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + f_1(t) \\ y_2 &= a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + f_2(t) \\ &\vdots \\ y_n &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + f_2(t), \end{aligned}$$

where $a_{ij}(t)$ with $1 \leq i, j \leq n$ are given coefficient functions and $f_i(t)$ are given forcing functions. Then the linear system can be rewritten in the matrix form

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix} \quad (1)$$

- If we use substitutions, then we can write (1) in more compact form:

$$\frac{d}{dt}\boldsymbol{y} = \boldsymbol{A}(t)\boldsymbol{y} + \boldsymbol{f}(t), \tag{2}$$

where

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \boldsymbol{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \end{bmatrix}, \boldsymbol{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

- We can also combine the linear system of DEs in (2) with the initial data to consider an IVP:

$$\frac{d}{dt}\boldsymbol{y} = \boldsymbol{A}(t)\boldsymbol{y} + \boldsymbol{f}(t), \quad \boldsymbol{y}(t_0) = \boldsymbol{k}, \tag{3}$$

where $t_0$ is an initial time and $\boldsymbol{k}$ is an initial data given by

$$\boldsymbol{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}.$$

**Theorem**

*(10.2.1) Suppose that $a_{ij} \in C[a, b]$ in $\boldsymbol{A}(t)$ and $f_i \in C[a, b]$ and $t_0 \in (a, b)$ and $\boldsymbol{k} \in \mathbb{R}^n$. Then $\exists! \boldsymbol{y}$ on $(a, b)$ satisfying (3).*